THE TWO-SECTOR OVERLAPPING GENERATIONS MODEL: A SIMPLE FORMULATION

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Abstract: In this paper we provide a simple formulation of a two-sector overlapping generations model based on the social production function which characterizes the factor-price frontier associated with interior temporary equilibria. We first give a simple restriction for the existence of a long run equilibrium. Under gross substitutability, we show that such a steady state is locally indeterminate if the consumption good is sufficiently capital intensive, the elasticity of savings with respect to wage is low enough and the elasticity of savings with respect to the interest rate is large enough. We also exhibit a flip bifurcation giving rise to period-two cycles. When gross substitutability is violated, we prove that local indeterminacy may also occur when the investment good is capital intensive. However, when the consumption good is capital intensive, it requires the existence of at least two distinct steady state. Indeed saddle-point stability is obtained as soon as uniqueness of the steady state holds. We also give conditions for the existence of complex characteristic roots and we exhibit a Hopf bifurcation giving rise to quasi-periodic cycles.

Keywords: Two-sector overlapping generations model, social production function, indeterminacy, endogenous fluctuations, periodic and quasi-periodic cycles.

JEL Classification Numbers: C62, E32, O41
1 Introduction

The existence of endogenous fluctuations and local indeterminacy of equilibria in two-sector overlapping generations (OLG) models is a well-established fact since the contributions of Galor [16] and Reichlin [25]. These results complement and generalize a huge number of papers dealing with the aggregate Diamond [14] model. Reichlin [25] shows the possibility of periodic, quasi-periodic and chaotic dynamics in a two-sector model with one pure consumption good and one investment good, both produced with Leontief technologies. Galor [16] considers on the contrary a general formulation with non-zero factors substitution in production but focuses on the existence of periodic cycles and local indeterminacy of equilibria.

Galor [16] considers an OLG counterpart to the two-sector optimal growth model formulated by Uzawa [26, 27]. His stock/price formulation is general enough to consider specialized economies in which only one good is produced and leads to a global characterization of the dynamical properties. However, it makes difficult to compare with the literature on optimal growth. It is well-known indeed since Benhabib and Nishimura [3, 4], Boldrin and Montrucchio [8], that optimal paths in two-sector optimal growth models may also exhibit cyclical or chaotic fluctuations. All these results have been established with models formulated in terms of a social production function which characterizes the factor-price frontier associated with interior temporary equilibria. Following Burmeister et al. [12], such a formulation has became the standard way to analyze multisector optimal growth models.

Our main objective in this paper is to develop a similar formulation for two-sector OLG models and to provide a simple framework in which OLG and optimal growth models may be directly compared. We will thus provide simple conditions in terms of the capital intensity difference across sectors and the elasticities of the saving function for the existence of endogenous fluctuations and local indeterminacy of equilibria. Our formulation will allow us to get slightly more precise conclusions than in Galor [16] and to provide direct economic interpretations of the results. We first give a simple restriction for the existence of a long run equilibrium. Under gross substitutability, we show that such a steady state is locally indeterminate.

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1See for instance Grandmont [17], Reichlin [24], Farmer [15], Jullien [18], Benhabib and Laroque [2] and more recently de la Croix and Michel [13].

2See the examples provided by Azariadis [1] and Ralf [23]. Kalra [19] also provides an extension to a two-sector model in which both goods enter consumption.

3See also Boldrin [6], Boldrin and Deneckere [7].

4See for instance Burmeister and Kuga [11], Brock [10], Kuga [20].
if the consumption good is sufficiently capital intensive, the elasticity of savings with respect to wage is low enough and the elasticity of savings with respect to the interest rate is large enough. We also exhibit a flip bifurcation giving rise to period-two cycles. When gross substitutability is violated, we prove that local indeterminacy may also occur when the investment good is capital intensive. However, when the consumption good is capital intensive, it requires the existence of at least two distinct steady state. Indeed saddle-point stability is obtained as soon as uniqueness of the steady state holds. We also give conditions for the existence of complex characteristic roots and we exhibit a Hopf bifurcation giving rise to quasi-periodic cycles.

This paper is organized as follows: Section 2 presents the basic model and the social production function formulation. In Section 3 we study the existence of steady state and we provide the expression of the characteristic polynomial. Section 4 contains the main results under gross substitutability. The local dynamical analysis when gross substitutability is violated is provided in Section 5. Section 6 finally contains a comparison of our results with related literature.

2 The model

2.1 Production

There are two commodities in the economy with one pure consumption good $y_0$ and one capital good $y$. Each good is produced with a standard constant returns to scale technology:

$$y_0 = f^0(k_0, l_0), \quad y = f^1(k_1, l_1)$$

with $k_0 + k_1 \leq k$, $k$ being the total stock of capital, and $l_0 + l_1 \leq \ell$, $\ell$ being the total amount of labor.

**Assumption 1.** Each production function $f^i : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$, $i = 0, 1$, is $C^2$, increasing in each argument, concave, homogeneous of degree one and such that for any $x > 0$, $f^1_1(0, x) = f^1_2(x, 0) = +\infty$, $f^1_1(+\infty, x) = f^1_2(x, +\infty) = 0$.

For any given $(k, y, \ell)$, we define a temporary equilibrium by solving the following problem of optimal allocation of factors between the two sectors:

$$\max_{k_0, k_1, l_0, l_1} f^0(k_0, l_0)$$

s.t. \begin{align*}
y &\leq f^1(k_1, l_1) \\
k_0 + k_1 &\leq k \\
l_0 + l_1 &\leq \ell \\
k_0, k_1, l_0, l_1 &\geq 0
\end{align*} \quad (1)
The associated Lagrangian is

\[ L = f^0(k_0, l_0) + p[f^1(k_1, l_1) - y] + r[k - k_0 - k_1] + w[\ell - l_0 - l_1] \]

with \( p \) the price of the investment good, \( r \) the rental rate of capital and \( w \) the wage rate, all in terms of the price of the consumption good. Solving the associated first order conditions gives optimal demand functions for capital and labor, namely \( k_0(k, y, \ell) \), \( l_0(k, y, \ell) \), \( k_1(k, y, \ell) \) and \( l_1(k, y, \ell) \). The resulting value function

\[ T(k, y, \ell) = f^0(k_0(k, y, \ell), l_0(k, y, \ell)) \]

is called the social production function and describes the frontier of the production possibility set. Constant returns to scale of technologies imply that \( T(k, y, \ell) \) is homogeneous of degree one and concave. We will assume in the following that \( T(k, y, \ell) \) is at least \( C^2 \).

We also get from the first order conditions

\[
\begin{align*}
r(k, y, \ell) &= f^0_0(k_0(k, y, \ell), l_0(k, y, \ell)) = p f^1_1(k_1(k, y, \ell), l_1(k, y, \ell)) \\
w(k, y, \ell) &= f^0_2(k_0(k, y, \ell), l_0(k, y, \ell)) = p f^1_2(k_1(k, y, \ell), l_1(k, y, \ell)) \\
p(k, y, \ell) &= f^0_0(k_0(k, y, \ell), l_0(k, y, \ell)) = f^0_2(k_0(k, y, \ell), l_0(k, y, \ell)) \\
&= f^0_0(k_0(k, y, \ell), l_0(k, y, \ell)) = f^0_2(k_0(k, y, \ell), l_0(k, y, \ell)) = f^0_2(k_0(k, y, \ell), l_0(k, y, \ell))
\end{align*}
\]

(2)

and it is easy to show that the rental rate of capital, the price of investment good and the wage rate satisfy

\[
\begin{align*}
T_1(k, y, \ell) &= r(k, y, \ell), & T_2(k, y, \ell) &= -p(k, y, \ell), & T_3(k, y, \ell) &= w(k, y, \ell)
\end{align*}
\]

(3)

2.2 Consumption and savings

The economy is populated by finitely-lived agents. In each period \( t \), \( N_t \) persons are born, and they live for two periods. In their first period of life (when young), the agents are endowed with one unit of labor that they supply inelastically to firms. Their income is equal to the real wage. They allocate this income between current consumption and savings which are invested in the firms. In their second period of life (when old), they are retired. Their income is given by the return on the savings made at time \( t \). As they do not care about events occurring after their death, they consume their income entirely. The preferences of a representative agent born at time \( t \) are thus defined over his consumption bundle \( (c_t, d_{t+1}) \), when he is young, and \( d_{t+1} \), when he is old) and are summarized by the utility function \( u(c_t, d_{t+1}) \).

Assumption 2. \( u(c, d) \) is increasing with respect to each argument \( (u_1(c, d) > 0 \) and \( u_2(c, d) > 0) \), \( C^2 \), with negative definite Hessian matrix, over the interior of \( \mathbb{R}^2_+ \). Moreover, for all consumption levels \( c,d > 0 \), \( u_1(0, d) = u_2(c, 0) = \infty \).
Each agent is assumed to have one child so that population is constant, i.e. \( N_t = N \). Considering the wage rate \( w_t \) and the expected gross rate of return on financial assets \( R_{t+1}^e \) as given, he maximizes his utility function over his life-cycle as follows:

\[
\max_{c_t, d_{t+1}} u(c_t, d_{t+1}) \\
\text{s.t.} \quad w_t = c_t + s_t \\
R_{t+1}^e s_t = d_{t+1}
\]  

(4)

The first order condition easily derives as

\[
\frac{u_1(c_t, d_{t+1})}{u_2(c_t, d_{t+1})} = R_{t+1}^e
\]

Agents perfectly expect the gross return to savings \( R_{t+1}^e = R_t^e + \) Assumption 2 implies the existence and uniqueness of an interior solution for optimal savings

\[
s_t = s(w_t, R_{t+1})
\]

We also introduce the following standard restriction:

**Assumption 3.** Second period consumption \( d \) is a normal good.

It is easy to prove that under Assumptions 2-3, the saving function \( s(., .) \) is differentiable for \((w, R) \in \mathbb{R}_+^2 \) and increasing with respect to the wage rate, i.e. \( s_1(w, R) \geq 0 \). Concerning the effect of the gross return to savings, we assume that consumption in both periods are gross substitute, i.e. the substitution effect following an increase in the gross return is greater (in absolute value) than the income effect:

**Assumption 4.** For any \((w, R) > 0 \), \( s_2(w, R) > 0 \).

### 2.3 Perfect foresight equilibrium

Under complete depreciation within one period, the capital accumulation equation is

\[
k_{t+1} = y_t
\]

(5)

At each time \( t \) total consumption is given by the social production function where total labor is given by the number \( N \) of young households, i.e. \( \ell = N \). It follows that \( \ell(c_t + d_t) = T(k_t, y_t, \ell) \). Along a dynamic competitive equilibrium, the production of the investment good equals total savings such that

\[
p_t y_t = \ell s(w_t, R_{t+1})
\]

(6)

The price of investment good is given by (3) while the gross rate of return on financial assets \( R_{t+1}^e \) is obtained as follows: starting from the equality
\(\ell(c_{t+1} + d_{t+1}) = T(k_{t+1}, y_{t+1}, \ell)\) and using the budget constraints of the representative agent with the homogeneity of the social production function we get

\[\ell(w_{t+1} - s_{t+1} + R_{t+1}s_t) = r_{t+1}k_{t+1} - pt+1yt+1 + w_{t+1}\ell\]
\[\iff R_{t+1}p_t k_{t+1} = r_{t+1}k_{t+1}\]

It follows that an equilibrium path satisfies the following difference equation of order two:

\[T_2(k_t, k_{t+1}, \ell)k_{t+1} + s \left( T_3(k_t, k_{t+1}, \ell), \frac{T_1(k_{t+1}, k_{t+2}, \ell)}{T_2(k_t, k_{t+1}, \ell)} \right) = 0 \quad (7)\]

Equation (7) defines an implicit two-dimensional dynamical system.

### 3 Linearized dynamical system

#### 3.1 Steady state

A steady state is defined as \(k_t = k^*, y_t = y^* = k^*, p_t = p^* = -T_2(k^*, k^*, \ell),\)
\(r_t = r^* = T_1(k^*, k^*, \ell), w_t = w^* = T_3(k^*, k^*, \ell)\) and \(R^* = r^*/p^*\) for all \(t\) with \(\ell = N\). Recall now that \(T(k, y, \ell)\) is a linear homogeneous function. This property is based on the fact that the capital and labor demand functions \(k_0(k, y, \ell), l_0(k, y, \ell), k_1(k, y, \ell)\) and \(l_1(k, y, \ell)\) are homogeneous of degree 1.

Then denoting \(\kappa = k/\ell\), a steady state \(k^*\) may be also defined as a \(\kappa^*\) solution of the following equation

\[T_2(\kappa, \kappa, 1)\kappa + s \left( T_3(\kappa, \kappa, 1), -T_1(\kappa, \kappa, 1)/T_2(\kappa, \kappa, 1) \right) = 0 \quad (8)\]

The following proposition provides a sufficient condition for the existence of a steady state

**Proposition 1.** Under Assumptions 1-3, there exists a steady state \(\kappa^* = k^*/\ell > 0\) solution of equation (8) if

\[\lim_{\kappa \to 0} s \left( T_3(\kappa, \kappa, 1), -T_1(\kappa, \kappa, 1)/T_2(\kappa, \kappa, 1) \right) > -\lim_{\kappa \to 0} T_2(\kappa, \kappa, 1)\kappa\]

#### 3.2 Characteristic polynomial

In order to derive a tractable formulation for the characteristic polynomial, we need to compute the second derivatives of \(T(k, y, \ell)\). As already mentioned above, we know that \(T(k, y, \ell)\) is a concave function. We have in particular:
\( T_{11}(k, y, \ell) = \frac{\partial r}{\partial k} \leq 0, \quad T_{22}(k, y, \ell) = -\frac{\partial p}{\partial y} \leq 0, \quad T_{33}(k, y, \ell) = \frac{\partial w}{\partial \ell} \leq 0 \)

However the sign of the cross derivatives is not obvious. To study these derivatives we start from the homogeneity property of the production functions. We have from (2):

\[ 1 = \frac{k_{0}}{k_{0}} r + \frac{k_{0}}{y_{0}} w, \quad 1 = \frac{k_{1}}{y_{1}} p + \frac{w}{y} p \]

which is equivalent to

\[ \begin{pmatrix} w & r \end{pmatrix} \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} = \begin{pmatrix} 1 & p \end{pmatrix} \tag{9} \]

with

\[ a_{00} = l_{0}/y_{0}, \quad a_{10} = k_{0}/y_{0}, \quad a_{01} = l_{1}/y, \quad a_{11} = k_{1}/y \]

the capital and labor coefficients in each sector. Equation (9) gives the factor-price frontier and corresponds to the equality between price and cost. Differentiating this equation gives:

\[ \begin{pmatrix} dw & dr \end{pmatrix} \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} + \begin{pmatrix} w & r \end{pmatrix} \begin{pmatrix} da_{00} & da_{01} \\ da_{10} & da_{11} \end{pmatrix} = \begin{pmatrix} 0 & dp \end{pmatrix} \]

It can be easily shown that the envelope theorem implies

\[ \begin{pmatrix} w & r \end{pmatrix} \begin{pmatrix} da_{00} & da_{01} \\ da_{10} & da_{11} \end{pmatrix} = 0 \]

so that

\[ \begin{pmatrix} dw & dr \end{pmatrix} \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} = \begin{pmatrix} 0 & dp \end{pmatrix} \]

Eliminating \( dw \) we can solve this system to get

\[ \frac{dp}{dr} = a_{01} \left( \frac{a_{11}}{a_{01}} - \frac{a_{10}}{a_{00}} \right) \equiv b \tag{10} \]

\( b \) is a relative capital intensity difference. The sign of \( b \) is thus positive if and only if the investment good is capital intensive. We can also solve the above system by eliminating \( dr \) and get

\[ \frac{dw}{dp} = -\frac{a_{10}}{a_{00}} b^{-1} \equiv -ab^{-1} \tag{11} \]

with \( a = a_{10}/a_{00} = k_{0}/l_{0} > 0 \) the capital-labor ratio in the consumption good sector. Now consider the cross derivatives. We can write:

\[ T_{12} = -\frac{\partial p}{\partial r} = -T_{11}b \]

\[ T_{31} = \frac{\partial w}{\partial p} \frac{\partial p}{\partial k} = -\frac{\partial w}{\partial p} T_{12} = -T_{11}a \geq 0 \]

\[ T_{32} = \frac{\partial w}{\partial p} \frac{\partial p}{\partial y} = \frac{\partial w}{\partial p} \frac{\partial r}{\partial y} = \frac{\partial w}{\partial p} b T_{12} = T_{11}ab \]
As already shown by Benhabib and Nishimura (1985), the sign of $T_{12}(k, y, \ell)$ is given by the sign of the relative capital intensity difference between the two sectors $b$. We show here that $T_{31}(k, y, \ell)$ is always positive while the sign of $T_{32}(k, y, \ell)$ is given by the sign of $-b$. Notice also that $T_{22}(k, y, \ell)$ and $T_{33}(k, y, \ell)$ may be written as

$$T_{22} = \frac{\partial p}{\partial r} \frac{\partial c}{\partial y} = T_{11} b^2$$

$$T_{33} = \frac{\partial w}{\partial p} \frac{\partial p}{\partial \ell} = \frac{\partial w}{\partial p} \frac{\partial w}{\partial y} = T_{11} a^2$$

Remark: The derivative $dr/dp$ is well-known in trade theory as the Stolper-Samuelson effect. It follows from the above computations that $dr/dp = b^{-1}$

Considering also the full employment equation derived from the stock constraints in program (1), we get the factor market clearing equation

$$\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \begin{pmatrix} y_0 \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ k \end{pmatrix}$$

At constant prices, the input coefficients remain fixed and we obtain the associated Rybczinsky effect

$$\frac{dy}{dk} = b^{-1}$$

We therefore find the well-known duality between the Rybczinsky and Stolper-Samuelson effects.

Based on the above computations, the characteristic polynomial is finally derived from total differentiation of equation (7). Denoting $s(w^*, r^*) = s^*$, $s_i(w^*, r^*) = s_i^*$, $T_i(k^*, k^*, \ell) = T_i^*$, $T_{ij}(k^*, k^*, \ell) = T_{ij}^*$, $i, j = 1, 2$, we get:

$$P(\lambda) = \lambda^2 \left[ \ell s_1^* T_1^* \ell s_2^* T_2^* b + \lambda \left( T_2^* + T_1^* \left[ b^2 k^* + \ell s_1^* a - \ell s_2^* \left( 1 - b^2 T_1^* T_2^* \right) \right] \right) \right] - T_1^* \left( b k^* + \ell s_1^* a + \ell s_2^* \frac{T_2^*}{T_1^*} \right) = 0$$

Remark: Notice that if $b = 0$ we get a one-sector model à la Diamond [14] with a degree-one characteristic polynomial. Similarly, if savings are independent of the gross rate of return on financial assets, i.e. $s_2(w, R) = 0$, the dynamical system is also one-dimensional.

Before analyzing the local properties of equilibrium paths, we will write the characteristic polynomial in terms of standard elasticities in order to get economically interpretable conditions. We define indeed the elasticities of the saving function with respect to wage and gross return
\[
\varepsilon_{sw} = s_1^* T_3^* / s^*, \quad \varepsilon_{sR} = -s_2^* T_1^* / (T_2^* s^*)
\]

the elasticity of the interest rate with respect to the capital stock

\[
\varepsilon_{rk} = -T_{11}^* k^* / T_1^*
\]

the elasticity of the price of investment good with respect to its output

\[
\varepsilon_{py} = T_{22}^* k^* / T_2^*
\]

and the elasticity of the wage rate with respect to labor

\[
\varepsilon_{w\ell} = -T_{33}^* \ell / T_3^*
\]

Using also the fact that linear homogeneity of the social production function implies at the steady state \(a\ell = k^*(1-b)\),\(^5\) and assuming that \(b \neq 0\), we easily obtain the following expression for the characteristic polynomial\(^6\)

\[
\mathcal{P}(\lambda) = \lambda^2 b^2 \varepsilon_{sR} \varepsilon_{rk} - \lambda b \left\{1 + \varepsilon_{sR} \varepsilon_{rk} + \varepsilon_{py} (1 + \varepsilon_{sR}) + \varepsilon_{sw} \varepsilon_{w\ell} \frac{b}{1-b}\right\}
\]

\[
+ \varepsilon_{py} (1 + \varepsilon_{sR}) + \varepsilon_{sw} \varepsilon_{w\ell} \frac{b}{1-b} = 0
\]

### 4 Main results

As a preliminary result, we show that under Assumption 4, the characteristic roots cannot be complex

**Lemma 1.** **Under Assumptions 1-4, the characteristic roots are real.**

Following the same methodology as in the two-sector optimal growth model, we discuss the local stability properties of equilibrium paths depending on the sign of the capital intensity difference across sectors \(b\). In order to simplify the formulations, we will give the main conditions in terms of the elasticity of savings with respect to the wage rate. We first define the following critical values for \(\varepsilon_{sw}\):

\[
\epsilon_1 = \frac{1}{\varepsilon_{w\ell}} - \frac{1-b}{b \varepsilon_{w\ell}} \varepsilon_{py} + \varepsilon_{sR} \varepsilon_{rk}
\]

\[
\epsilon_2 = -\frac{1-b}{b} \varepsilon_{sw} \varepsilon_{w\ell} \frac{1}{1-b} \varepsilon_{w\ell}
\]

which allows to determine the sign of \(\mathcal{P}(1)\) and \(\mathcal{P}(-1)\) and thus to locate the characteristic roots with respect to the unit circle.

\(^5\)Linear homogeneity of \(T(k, y, \ell)\) implies that the first derivatives \(T_i(k, y, \ell)\) are homogeneous of degree 0. Using the previous computations of the second order derivatives of \(T\) in terms of capital and labor coefficients we get at the steady state: \(0 = T_{i1}^* k^* + T_{i2}^* y^* + T_{i3}^* \ell = T_{i1}^*[1 - b)k^* - a\ell].\)

\(^6\)Simple computations give indeed the following expressions: \(T_{s1}^* = -\varepsilon_{sw} T_3^* k^* / T_3^*,\)

\(T_{s2}^*/T_2^* = \varepsilon_{sR} T_2^* k^* / T_2^*,\)

\(T_{s3}^*/T_3^* = \varepsilon_{rk} b^2 k^*/(\varepsilon_{py a^2 \ell^2}),\)

\(T_{s2}^*/T_1^* = -\varepsilon_{sw} b^2 / \varepsilon_{py}.\) The new formulation for \(\mathcal{P}(\lambda)\) is then directly obtained from substitution of all this into (12).
4.1 A capital intensive investment good

In a two-sector optimal growth model, Benhabib and Nishimura [4] show that when the investment good is capital intensive, i.e. \( b > 0 \), the optimal path monotonically converges toward the steady state. We find a similar conclusion under some additional restriction on the saving function.

**Proposition 2.** Under Assumptions 1-4, let the investment good be capital intensive. Then the equilibrium path is monotone and any steady-state \( k^* \) is either a saddle-point or totally unstable. Moreover, saddle-point stability holds for a given steady state \( k^* \) iff \( \varepsilon_{sw} < \epsilon_1 \).

Saddle-point stability is then obtained if the elasticity of savings with respect to the wage rate is low enough. Notice that this condition may be equivalently stated on the basis of the elasticity of savings with respect to the gross rate of return as follows

\[
\varepsilon_{sR} > \frac{\varepsilon_{py}}{b\varepsilon_{rk} - \varepsilon_{py}} - \frac{b_{R}}{b_{R} - \varepsilon_{py}} (1 - \varepsilon_{sw} \varepsilon_{sw})
\]

We then get a kind of complementary condition for the elasticity of savings with respect to the gross rate of return which needs to be high enough.

Proposition 2 provides a necessary and sufficient condition for saddle-point stability when multiple steady states arise. However, it is easy to derive from Proposition 1 that if there exists a unique steady state, saddle-point stability always holds and the same conclusion as in the two-sector optimal growth model occurs.

**Corollary 1.** Under Assumptions 1-4, let the investment good be capital intensive. If there exists a unique steady-state \( k^* \), then it is saddle-point stable and the equilibrium path monotonically converges to \( k^* \).

4.2 A capital intensive consumption good

When the consumption good is capital intensive, we show that local indeterminacy of equilibria may occur. This result echoes the conclusions of Benhabib and Nishimura [5] who show in a two-sector infinite-lived agent model with sector specific externalities that multiple equilibria also require a capital intensive consumption good at the private level.

**Proposition 3.** Under Assumptions 1-4, let the consumption good be capital intensive.

i) When \( b > -1 \), any steady state \( k^* \) is locally determinate. Moreover,
a given $k^*$ is saddle-point stable iff $\varepsilon_{sw} \in (\varepsilon_2, \varepsilon_1)$.

ii) When $b < -1$ and

$$\varepsilon_{sR} > \frac{1}{(1-b^2)\varepsilon_{rk}}$$

a given steady state $k^*$ is locally indeterminate iff $\varepsilon_{sw} \in (\varepsilon_2, \varepsilon_1)$ and saddle-point stable iff $\varepsilon_{sw} \in (0, \varepsilon_2) \cup (\varepsilon_1, +\infty)$.

iii) When $b < -1$ and

$$\varepsilon_{sR} < \frac{1}{(1-b^2)\varepsilon_{rk}}$$

any steady state $k^*$ is locally determinate. Moreover, a given $k^*$ is saddle-point stable iff $\varepsilon_{sw} \in (0, \varepsilon_1) \cup (\varepsilon_2, +\infty)$.

Notice that, for the occurrence of local indeterminacy, we find again a kind of complementary relationship between the elasticities of savings with respect to the gross rate of return and the wage rate: $\varepsilon_{sw}$ needs to be low enough (but not too small) while $\varepsilon_{sR}$ needs to be high enough.

In Proposition 3 we do not explicitly discuss the existence of negative characteristic roots. This point is important since endogenous fluctuations through a flip bifurcation may occur. The following Corollary provides some results.

**Corollary 2.** Under Assumptions 1-4, if the consumption good is capital intensive, then the equilibrium path is oscillating and a flip bifurcation may occur when $\varepsilon_{sw}$ crosses $\varepsilon_2$ from above.

As in the two-sector optimal growth model, endogenous fluctuations require a capital intensive consumption good. Following Benhabib and Nishimura [4], we can use the Rybczinski and Stolper-Samuelson effects to provide a simple economic intuition for this result. Assume that the consumption good is capital intensive, i.e. $b < 0$, and consider an instantaneous increase in the capital stock $k_t$. This results in two opposing forces:

- Since the consumption good is more capital intensive than the investment good, the trade-off in production becomes more favorable to the consumption good. Moreover, the Rybczinski effect implies a decrease of the output of the capital good $y_t$. This tends to lower the investment and the capital stock in the next period $k_{t+1}$.

- In the next period the decrease of $k_{t+1}$ implies again through the Rybczinski effect an increase of the output of the capital good $y_{t+1}$. This mechanism is explained by the fact that the decrease of $k_{t+1}$ improves the trade-off in production in favor of the investment good which is relatively less intensive in capital. Therefore this tends to increase the investment and
the capital stock in period $t + 2$, $k_{t+2}$. Notice also that the rise of $y_{t+1}$ implies a decrease of the rental rate $r_{t+1}$ and through the Stolper-Samuelson effect an increase of the price $p_{t+1}$.

So far the above discussion concerns the existence of oscillations but not that of persistent cycles. For cycles to be sustained, the oscillations in relative prices must not present intertemporal arbitrage opportunities. Thus possible gains from postponing consumption from periods when the marginal rate of transformation between consumption and investment is high to periods when it is low must not be worth it. In OLG models, whether this is the case or not depends on the properties of the saving function. The instantaneous increase in the capital stock $k_t$ implies an increase of the wage rate $w_t$ and, through the Rybczinsky effect, an increase of the consumption of young agents born at time $t$. It follows that a decrease of their consumption when old, i.e., at time $t + 1$, will require a low enough elasticity of their savings with respect to the wage rate which is not enough to compensate the decrease of the production of the consumption good at time $t + 1$ generated by the Rybczinsky effect.

Notice that in two-sector optimal growth models, similar additional conditions are introduced but they directly concern the preferences: a minimum level of myopia, i.e. a low enough value for the discount rate $\delta$, is necessary and the elasticity of intertemporal substitution in consumption needs to be large enough to allow the consumer to compensate some decrease of current consumption by some increase of future consumption. However more demanding restrictions on the capital intensity difference need to be introduced, i.e. $b \in (-1, -\delta)$ with $\delta \in (0,1)$ (see Benhabib and Nishimura [4]).

5 Extensions

All our main results have been obtained under Assumption 4 which requires a strictly increasing saving function with respect to the gross rate of return. Two extensions covering the converse configuration may be considered.

5.1 $s_2(w, R) = 0$

In this particular case, savings are independent of the gross rate of return on financial assets. It follows that $\varepsilon_s R = 0$, the dynamical system is one-dimensional and the characteristic polynomial becomes

\[ \text{See Bosi, Magris and Venditti [9], Nishimura, Takahashi and Venditti [21].} \]
\[ P(\lambda) = -\lambda b \left( 1 + \epsilon_{py} + \epsilon_{sw} \epsilon_{wl} \frac{b}{1-b} \right) + \epsilon_{py} + \epsilon_{sw} \epsilon_{wl} \frac{b}{1-b} \]

In such a case, the equilibrium is locally determinate and local stability will be obtained if and only if \(-1 < \lambda < 1\), or equivalently \(P(1)P(-1) < 0\). Consider the critical bounds (13) with \(\epsilon_{sR} = 0\). Then we get:

**Proposition 4.** Under Assumptions 1-3, let \(s_2(w,R) = 0\). Then, any steady-state \(k^*\) is locally determinate. More precisely, a given \(k^*\) is saddle-point stable iff one of the following conditions is satisfied:

i) \(b > 0\) and \(\epsilon_{sw} < \epsilon_1\). Moreover, the equilibrium path is monotone.

ii) \(b > 0\) and \(k^*\) is unique. Moreover, the equilibrium path is monotone.

iii) \(b > -1\) and \(\epsilon_{sw} \in (\epsilon_2, \epsilon_1)\). Moreover, the equilibrium path is oscillating when \(\epsilon_{sw} \in (\epsilon_2, \epsilon_0)\), monotone when \(\epsilon_{sw} \in (\epsilon_0, \epsilon_1)\), and a flip bifurcation may occur when \(\epsilon_{sw}\) crosses \(\epsilon_2\) from above.

iv) \(b < -1\) and \(\epsilon_{sw} \in (0, \epsilon_1) \cup (\epsilon_2, +\infty)\). Moreover, the equilibrium path is oscillating when \(\epsilon_{sw} \in (0, \epsilon_0) \cup (\epsilon_2, +\infty)\), monotone when \(\epsilon_{sw} \in (\epsilon_0, \epsilon_1)\), and a flip bifurcation may occur when \(\epsilon_{sw}\) crosses \(\epsilon_2\) from above.

5.2 \(s_2(w,R) < 0\)

The standard assumption of an increasing saving function with respect to the gross rate of return implies that the characteristic roots are necessarily real. If we assume on the contrary that savings are monotonically decreasing in \(R\), Lemma 1 does not necessarily hold. Moreover, local indeterminacy of equilibria may also arise while the investment good is capital intensive, and when the characteristic roots are complex, a Hopf bifurcation may occur.

In order to discuss whether the characteristic roots are complex or real, and whether local indeterminacy may arise or not, we have to introduce two additional critical values for \(\epsilon_{sw}\):

\[
\tilde{\epsilon} = \frac{1-b \epsilon_{sR} \epsilon_{wk} + 2 \sqrt{\epsilon_{sR} \epsilon_{wk} - 1 - \epsilon_{py} (1 + \epsilon_{sR})}}{\epsilon_{wl}}
\]

\[
\epsilon_3 = \frac{-1-b \epsilon_{py} (1 + \epsilon_{sR}) - b^2 \epsilon_{sR} \epsilon_{wk}}{\epsilon_{wl}}
\]

(14)

The bound \(\tilde{\epsilon}\) allows indeed to determine the sign of the discriminant, while \(\epsilon_3\) allows to locate the determinant with respect to 1. We also consider the critical values (13) previously defined. We first show that contrary to the case with gross substitutability, local indeterminacy may now arise while the investment good is capital intensive.

**Proposition 5.** Under Assumptions 1-3, let \(s_2(w,R) < 0\) and the investment good be capital intensive.

1) When
a given steady state \( k^* \) is locally indeterminate iff \( \varepsilon_{sw} \in (\varepsilon_3, \varepsilon_2) \) and saddle-point stable iff \( \varepsilon_{sw} \in (\varepsilon_2, \varepsilon_1) \);

2) When

\[-\varepsilon_{sR} > \frac{1}{(1+b)^2 \varepsilon_{rk}}\]

any steady state \( k^* \) is locally determinate. Moreover a given \( k^* \) is saddle-point stable iff one of the following cases holds:

i) \( \frac{1}{(1+b)^2 \varepsilon_{rk}} < -\varepsilon_{sR} < \frac{1}{(1-b^2) \varepsilon_{rk}} \) and \( \varepsilon_{sw} \in (\varepsilon_2, \varepsilon_1) \);

ii) \(-\varepsilon_{sR} > \frac{1}{(1-b^2) \varepsilon_{rk}} \) and \( \varepsilon_{sw} \in (\varepsilon_1, \varepsilon_2) \).

This result echoes the conclusions of Nishimura and Venditti [22] who show in a two-sector infinite-lived agent model that when intersectoral externalities are considered, multiple equilibria may arise when the consumption good is capital intensive at the private level.

As shown in Galor [17], if savings decrease with the gross rate of return local indeterminacy may still arise when the consumption good is capital intensive. However, we get the surprising result that local indeterminacy cannot arise when the steady state is unique.

**Proposition 6.** Under Assumptions 1-3, let \( s_2(w, R) < 0 \) and the consumption good be capital intensive.

1) If there exists a unique steady state \( k^* \), then it is locally determinate.

2) Any steady state \( k^* \) is saddle-point stable iff one of the following conditions is satisfied:

i) \( b > -1 \) and \( \varepsilon_{sw} \in (\min\{\varepsilon_1, \varepsilon_2\}, \max\{\varepsilon_1, \varepsilon_2\}) \);

ii) \( b < -1 \) and \( \varepsilon_{sw} \in (0, \varepsilon_1) \cup (\varepsilon_2, +\infty) \).

In order to provide conditions for local indeterminacy, we thus need to introduce the following assumption:

**Assumption 5.** There exist at least two steady states solutions of equation (8).

**Proposition 7.** Under Assumptions 1-3 and 5, let \( s_2(w, R) < 0 \) and the consumption good be capital intensive. Then a given steady state \( k^* \) is locally indeterminate if and only if one of the following conditions is satisfied:

1) \( b > -1 \),

\[-\varepsilon_{sR} < \frac{1}{(1+b)^2 \varepsilon_{rk}} \quad \text{and} \quad \varepsilon_{sw} \in (\max\{\varepsilon_1, \varepsilon_2\}, \varepsilon_3) \];

2) \( b < -1 \),

\[-\varepsilon_{sR} > \frac{1}{(1+b)^2 \varepsilon_{rk}} \quad \text{and} \quad \varepsilon_{sw} \in (\varepsilon_3, \varepsilon_2) \].
and $\varepsilon_{sw} \in (\epsilon_1, \min\{\epsilon_2, \epsilon_3\})$.

In Propositions 5 and 7 we do not explicitely discuss the existence of complex characteristic roots. This point is important since if local indeterminacy coexists with a negative discriminant, quasi-periodic cycles may occur through a Hopf bifurcation. The following Corollary provides some results.

**Corollary 3.** Under Assumptions 1-3, let $s_2(w, R) < 0$. Then:

1) When the investment good is capital intensive with

$$-\varepsilon_{sR} > \frac{1}{(1+b)^2\varepsilon_{rk}},$$

the characteristic roots are real for any $\varepsilon_{sw} \in (\epsilon_3, \epsilon_2)$ and a Hopf bifurcation cannot occur.

2) When the consumption good is capital intensive with

$$\frac{1}{(1+b)^2\varepsilon_{rk}} > -\varepsilon_{sR} > \frac{1}{(1-b)^2\varepsilon_{rk}},$$

then $\tilde{\varepsilon} \in (\epsilon_1, \epsilon_3)$ so that the characteristic roots are complex for any $\varepsilon_{sw} \in (\tilde{\varepsilon}, \epsilon_3)$ and a Hopf bifurcation may occur when $\varepsilon_{sw}$ crosses $\epsilon_3$ from below.

We have noticed previously that in a two-sector infinite-lived agent model with intersectoral externalities, local indeterminacy may arise while the investment good is capital intensive. But, contrary to what is shown in this class of models by Nishimura and Venditti [22], we show in case 1) that a Hopf bifurcation cannot occur under such a capital intensity configuration. The occurrence of quasi-periodic cycles through a Hopf bifurcation indeed requires a capital intensive consumption good. Notice however that when $b > -1$, the conditions for local indeterminacy are enough to get the result while an additional restriction on the elasticity $\varepsilon_{sR}$ needs to be introduced when $b < -1$.

In an optimal growth framework, we know since Benhabib and Nishimura [3] that the existence of a Hopf bifurcation requires the consideration of at least three-sector economies with two distinct investment goods. We show here that within an OLG framework, two-sector economies may be characterized by periodic or quasi-periodic orbits. However, as shown in Venditti [28, 29], with infinite-lived agent models, Hopf bifurcations are associated with capital intensive investment goods while in OLG models a capital intensive consumption good is required.
6 Comparison with related literature

The main comparison has to be made with the paper by Galor [17] in which he considers an OLG counterpart to the two-sector optimal growth model formulated by Uzawa [26, 27]. The dynamical system which describes the equilibrium path is written in a stock/price space and allows to consider specialized economies in which only one good is produced. On the contrary, our formulation is based on the social production function which characterizes interior temporary equilibria such that both goods are produced. Notice however that Galor is mainly interested in the dynamical characterization of interior equilibria.

Under gross substitutability, Galor shows that local indeterminacy arises if the following conditions are satisfied:

i) the consumption good is capital intensive,
ii) the production technologies in the two sectors are relatively dissimilar,
iii) the income effect following a rise in the current price of the consumption good is sufficiently stronger than the substitution effect.

Our formulation allows to provide more precise conditions which directly echoes the standard analysis of two-sector optimal growth models. As proved in Proposition 3, we show that when the consumption good is capital intensive, local indeterminacy occurs if

a) the production technologies are such that the capital intensity difference across sectors satisfies at the steady state $b < -1$,

b) the elasticity of savings with respect to the wage rate satisfies $\varepsilon_{sw} \in (\varepsilon_2, \varepsilon_1)$,

c) the income effect following a rise in the gross rate of return is sufficiently stronger than the substitution effect so that the elasticity of savings with respect to $R$ satisfies

$$\varepsilon_{SR} > \frac{1}{(1-b^2)r_k}$$

Condition a) gives indications on how the production technologies in the two sectors have to be dissimilar. Conditions b) and c) provide clear restrictions on the saving function, i.e. on preferences. In particular, condition c) evaluates how stronger the income effect needs to be with respect to the substitution effect. Moreover, we provide a bifurcation analysis by showing in Corollary 2 that the equilibrium path is oscillating and $\varepsilon_2$ may be a flip bifurcation value giving rise to persistent endogenous cycles.

When gross substitutability is not assumed and the saving function decreases with the gross rate of return, Galor shows that local indeterminacy

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8 See Benhabib and Nishimura [4, 5].
may now arise when the investment good is capital intensive. He also men-
tions that the steady state may be a spirale source and that periodic orbits
may occur. However he does not give any precise conditions for the occur-
rence of complex eigenvalues and does not provide a bifurcation analysis.
Our formulation allows to get a clear picture of the dynamical properties
of equilibrium paths. We first prove the strong result that when the con-
sumption good is capital intensive, local indeterminacy cannot arise if the
steady state is unique. We also show that local indeterminacy with complex
eigenvalues may only occur if the consumption good is capital intensive and
the income effect following a rise in the gross rate of return is not too strong
with respect to the substitution effect. In such a case we compute the Hopf
bifurcation value for the elasticity of savings with respect to the wage rate.

Our main conclusions may be also compared with the contribution of
Kalra [19] in which a two-sector economy with two consumption goods is
considered. There are indeed one pure consumption and one mixed good
which is used as consumption and capital. The two goods are consumed
by the representative agent during his both periods of life. They enter
the utility function through some consumption indices given by homothetic
functions. Such a formulation allows for the existence of intratemporal sub-
stitution in consumption. The analysis of Kalra is based on the notion of
critical economies initially introduced by Benhabib and Laroque [2] to study
a one-sector OLG model with production, money and endogenous labor sup-
ply. Critical economies are characterized by equilibria which are close but
different. This concept allows therefore to provide a bifurcation analysis.9
Notice however that Kalra does not study the existence of a steady state
but assumes it. He then shows that large elasticities of substitution in pro-
duction need not necessarily reduce the scope for cyclical orbits as long as
there are compensating changes on the consumption side: an increase in
substitution in production can be accompanied by a fall or rise in the in-
tertemporal substitution but must be accompanied by an attendant decline
in the intratemporal substitution. As in the standard framework with one
consumption good, endogenous fluctuations through flip bifurcations require
a capital intensive consumption good. However, Kalra also provides an ex-
ample in which a Hopf bifurcation may occur while the investment good is
capital intensive as long as the elasticity of capital labor substitution and
the intratemporal substitution are very close to zero.

9 A critical economy is defined with respect to configurations in which the modulus of
a characteristic root is equal to one.
7 Appendix

7.1 Proof of Proposition 1

From the definition of $T(k, y, \ell)$, it is easy to notice that $T(k, y, \ell) = 0$ if and only if $y_t = f^1(k_t, \ell)$. Moreover, under Assumption 1, there exists $\bar{k} > 0$, solution of $f^1(\bar{k}, \ell) = \bar{k}$, such that $f^1(k_t, \ell) > k_t$ if $k_t < \bar{k}$ while $f^1(k_t, \ell) < k_t$ if $k_t > \bar{k}$. Since $T(k, y, \ell)$ is homogeneous of degree one, we get along a stationary position

$$T(\kappa, \kappa, 1) = T_1(\kappa, \kappa, 1) + T_2(\kappa, \kappa, 1)\kappa + T_3(\kappa, \kappa, 1)\kappa$$

Now consider equation (8). From the first period budget constraint of households, we know that $s_t \leq w_t = T_3(k_t, y_t, \ell)$. We then have

$$T_2(\kappa, \kappa, 1)\kappa + s \left( T_3(\kappa, \kappa, 1) - \frac{T_1(\kappa, \kappa, 1)}{T_2(\kappa, \kappa, 1)} \right) \leq T_2(\kappa, \kappa, 1)\kappa + T_3(\kappa, \kappa, 1)\kappa$$

$$= T(\kappa, \kappa, 1) - T_1(\kappa, \kappa, 1)\kappa$$

Considering $\bar{\kappa} = \frac{\bar{k}}{\ell}$, we get

$$\lim_{\kappa \to \bar{\kappa}} T_2(\kappa, \kappa, 1)\kappa + s \left( T_3(\kappa, \kappa, 1) - \frac{T_1(\kappa, \kappa, 1)}{T_2(\kappa, \kappa, 1)} \right) \leq -T_1(\bar{\kappa}, \bar{\kappa}, 1)\bar{\kappa} < 0$$

The result follows. \hfill \Box

7.2 Proof of Lemma 1

The discriminant of the characteristic polynomial satisfies:

$$\Delta = b^2 \left\{ 1 + \varepsilon_s R \varepsilon_r k + \varepsilon_{py} (1 + \varepsilon_s R) + \varepsilon_{sw} \varepsilon_w \ell \frac{b}{1-b} \right\}^2$$

$$- 4b^2 \varepsilon_s R \varepsilon_r k \left( \varepsilon_{py} (1 + \varepsilon_s R) + \varepsilon_{sw} \varepsilon_w \ell \frac{b}{1-b} \right)$$

$$= b^2 \left\{ \left[ 1 - \varepsilon_s R \varepsilon_r k + \varepsilon_{py} (1 + \varepsilon_s R) + \varepsilon_{sw} \varepsilon_w \ell \frac{b}{1-b} \right]^2 + 4 \varepsilon_s R \varepsilon_r k \right\}$$

From Assumption 4 we get $\Delta \geq 0$. \hfill \Box

7.3 Proof of Proposition 2

From the characteristic polynomial we derive:

$$\mathcal{P}(0) = \varepsilon_{py} (1 + \varepsilon_s R) + \varepsilon_{sw} \varepsilon_w \ell \frac{b}{1-b}$$

$$\mathcal{P}(1) = -b + (1-b) \left( \varepsilon_{py} + \varepsilon_{sw} \varepsilon_w \ell \frac{b}{1-b} + \varepsilon_s R (\varepsilon_{py} - b \varepsilon_r k) \right)$$

$$\mathcal{P}(-1) = b + (1+b) \left( \varepsilon_{py} (1 + \varepsilon_s R) + \varepsilon_{sw} \varepsilon_w \ell \frac{b}{1-b} + b \varepsilon_s R \varepsilon_r k \right)$$

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Moreover, the characteristic roots satisfy
\[ \lambda_1 \lambda_2 \equiv D = \frac{\varepsilon_{py}(1 + \varepsilon_{sR}) + \varepsilon_{sw}\varepsilon_{w}t \frac{b}{1-b}}{b^2 \varepsilon_{sR}\varepsilon_{rk}} \]
\[ \lambda_1 + \lambda_2 \equiv T = \frac{1 + \varepsilon_{py}(1 + \varepsilon_{sR}) + \varepsilon_{sw}\varepsilon_{w}t \frac{b}{1-b} + \varepsilon_{sR}\varepsilon_{rk}}{b \varepsilon_{sR}\varepsilon_{rk}} \]

Under assumptions 1-4 and \( b > 0 \), we get \( D > 0 \) and \( T > 0 \) which imply that both roots are positive. The equilibrium path is therefore monotone.

Considering now that a competitive equilibrium is characterized by a complete utilization of the available capital stock, we have \( a_{10y_0} + a_{11y} = \bar{k} \) (18). Moreover, the steady-state production of the investment good is given by \( y^* = \bar{k}^* \). We then get
\[ \bar{k}^* (1 - a_{11}) = a_{10y_0} > 0 \]

From the definition of the capital intensity difference across sectors we finally obtain
\[ 1 - b = \frac{a_{00}(1-a_{11}) + a_{10}a_{01}}{a_{00}} > 0 \]

We derive from this that \( P(-1) > 0 \). Since we also get \( P(0) > 0 \), we conclude that the steady state is a saddle-point if \( P(1) < 0 \), locally indeterminate or totally unstable if \( P(1) > 0 \). Assume then that \( P(1) > 0 \), i.e.
\[ \frac{\varepsilon_{py}(1 + \varepsilon_{sR}) + \varepsilon_{sw}\varepsilon_{w} \frac{b}{1-b}}{b \varepsilon_{sR}\varepsilon_{rk}} > 1 + \frac{1}{b(1-b)\varepsilon_{sR}\varepsilon_{rk}} \]

Substituting this inequality into the trace \( T \) and using again the fact that \( 1 - b > 0 \), we get
\[ T > \frac{1}{b} + 1 + \frac{1}{b(1-b)\varepsilon_{sR}\varepsilon_{rk}} > 2 \]

It follows therefore that any steady-state \( \bar{k}^* \) is either a saddle-point (when \( P(1) < 0 \)) or totally unstable (when \( P(1) > 0 \)). The condition for \( P(1) < 0 \) rewrites as
\[ \varepsilon_{py} - \frac{b}{1-b}(1 - \varepsilon_{sw}\varepsilon_{w}t) + \varepsilon_{sR}(\varepsilon_{py} - b\varepsilon_{rk}) < 0 \]
or equivalently \( \varepsilon_{sw} < \varepsilon_{1} \). Notice however from equation (9) that we get at the steady state \( (a_{01}/a_{00}) + r^*b = p^* \). We conclude from this equality that \( T_1^*b < -T_2^* \) and since \( b > 0 \) we get
\[ \varepsilon_{py} - b\varepsilon_{rk} = \frac{T_2^*k^*}{T_2} + b\frac{T_{11}^*k^*}{T_1} = bT_{11}^*k^* \left( \frac{b}{T_2} + \frac{1}{T_1} \right) < 0 \]

The condition may thus be also stated in terms of the elasticity of savings with respect to \( R \) as \( \varepsilon_{sR} > \left[ \varepsilon_{py} - b(1 - \varepsilon_{sw}\varepsilon_{w}t)/(1-b) \right]/(b\varepsilon_{rk} - \varepsilon_{py}) \). \( \square \)
7.4 Proof of Corollary 1

Recall from equation (8) that a steady state \( \kappa^* = k^*/\ell \) is a solution of
\[
\theta(\kappa) \equiv T_2(\kappa, \kappa, 1) + s(T_3(\kappa, \kappa, 1), -T_1(\kappa, \kappa, 1)/T_2(\kappa, \kappa, 1)) = 0
\]
Using the expressions of the second derivatives \( T_{ij} \) of the social production function given in section 3.2, we easily compute the first derivative of \( \theta(\kappa) \) as
\[
\frac{\theta'(\kappa)}{T_2} = -1 + \frac{T_1}{T_2}(1 - b) \left[ b\kappa + s_1 + \frac{s_2}{T_2} \left( 1 + \frac{T_1}{T_2} \right) \right]
\]
From footnote 6 we know that at the steady state:
\[
a = \kappa^*(1 - b), \quad s_1^* = \epsilon_{sw}\epsilon_{w}\epsilon_{l}k^*/(\epsilon_{py}(1 + \epsilon_{R}\epsilon_{r}\epsilon_{k})), \quad s_2^*/T_2 = -\epsilon_{py}/(\epsilon_{r}\epsilon_{k})^2, \quad T_1^*/T_2 = \epsilon_{py}/(\epsilon_{r}\epsilon_{k}^2).
\]
Substituting all this into the previous equation gives
\[
\frac{\theta'(\kappa^*)}{T_2} = \frac{P(1)}{b}
\]
We have shown in the proof of Proposition 1 that there exists \( \bar{\kappa} > 0 \) such that \( \theta(\bar{\kappa}) < 0 \). If there exists a unique steady state \( \kappa^* \) then \( \theta(0) > 0 \) and we necessarily have \( \theta'(\kappa^*) < 0 \). When \( b > 0 \) this is equivalent to \( P(1) < 0 \) so that \( \kappa^* \) is saddle-point stable.

7.5 Proof of Proposition 3

The critical values \( \epsilon_1 \) and \( \epsilon_2 \) defined in Section 3 are useful to study respectively the sign of \( P(1) \) and \( P(1^-) \). A third critical value needs to be introduced in order to compare the determinant \( D \) to 1, namely
\[
\epsilon_3 = \frac{1-b}{b} \frac{\epsilon_{py}(1+\epsilon_{R})-b^2\epsilon_{sR}\epsilon_{tk}}{\epsilon_{w}}
\]
We start by comparing these three values. Straightforward computations give
\[
\begin{align*}
\epsilon_1 - \epsilon_2 &= \frac{2b(1+b)\epsilon_{sR}\epsilon_{tk}}{(1+b)\epsilon_{w}} \quad (15) \\
\epsilon_2 - \epsilon_3 &= \frac{b(1+b)\epsilon_{sR}\epsilon_{tk}}{(1+b)\epsilon_{w}} \\
\epsilon_1 - \epsilon_3 &= \frac{1+(1+b)\epsilon_{sR}\epsilon_{tk}}{(1+b)\epsilon_{w}}
\end{align*}
\]
Since \( \lim_{\lambda \to \pm \infty} \mathcal{P}(\lambda) = +\infty \), local indeterminacy of equilibria requires \( \mathcal{P}(1) > 0 \), \( \mathcal{P}(1^-) > 0 \) and \( D < 1 \). We already derive \( \mathcal{P}(1) > 0 \iff \epsilon_{sw} < \epsilon_1 \).

i) Assume first \( b > -1 \). This implies \((1+b) > 0 \) and thus \( \epsilon_1 > \epsilon_3 > \epsilon_2 \). Moreover we easily get \( \mathcal{P}(1^-) > 0 \iff \epsilon_{sw} < \epsilon_2 \) and \( D < 1 \iff \epsilon_{sw} > \epsilon_3 \). These two inequalities cannot be simultaneously satisfied. Any steady state is therefore locally determinate. Moreover, saddle-point stability holds if and only if \( \mathcal{P}(1) > 0 \) and \( \mathcal{P}(1^-) < 0 \), i.e. \( \epsilon_{sw} \in (\epsilon_2, \epsilon_1) \).
Assume now $b < -1$. This implies $(1 + b) < 0$ and $\mathcal{P}(-1) > 0$ $\iff$ $\varepsilon_{sw} > \varepsilon_2$. Moreover, we have $\varepsilon_1 > \varepsilon_3$, $\varepsilon_2 > \varepsilon_3$ and $D < 1$ $\iff$ $\varepsilon_{sw} > \varepsilon_3$. A steady state $k^*$ is then locally indeterminate if and only if $\varepsilon_1 > \varepsilon_2$, i.e.

$$1 + (1 - b)(1 + b)\varepsilon_s R \varepsilon_{rk} < 0$$

and $\varepsilon_{sw} \in (\varepsilon_2, \varepsilon_1)$. On the contrary, when $\varepsilon_{sw} \in (0, \varepsilon_2) \cup (\varepsilon_1, +\infty)$, we get $\mathcal{P}(1)\mathcal{P}(-1) < 0$ and saddle-point stability is obtained.

When $b < -1$ and $1 + (1 - b)(1 + b)\varepsilon_s R \varepsilon_{rk} < 0$, local indeterminacy cannot occur. Moreover we get $\varepsilon_2 > \varepsilon_1 > \varepsilon_3$ and saddle-point stability is obtained if and only if $\varepsilon_{sw} \in (0, \varepsilon_1) \cup (\varepsilon_2, +\infty)$.

### 7.6 Proof of Corollary 2

In order to discuss the sign of the characteristic roots, we need to introduce two additional critical values for $\varepsilon_{sw}$:

$$\epsilon_{0} = \frac{-1-b}{b} \frac{\varepsilon_{sw}}{\varepsilon_{wl}} \epsilon_{1} = \frac{1+(1-b)^2}{b} \frac{\varepsilon_{sw}}{\varepsilon_{wl}} \epsilon_{2} = \frac{1+(1-b)\varepsilon_s R \varepsilon_{rk}}{b \varepsilon_{wl}}$$

The bound $\epsilon_{0}$ allows indeed to determine the sign of the determinant, while $\hat{\epsilon}$ allows to determine the sign of the trace. We also consider the critical values (13) previously defined. We easily get the following:

$$\epsilon_{3} = \frac{b(1+(1-b)\varepsilon_s R \varepsilon_{rk})}{(1+b) \varepsilon_{wl}}$$

Moreover, under $b < 0$, we have $D < 0$ if and only if $\varepsilon_{sw} > \epsilon_{0}$, and $T < 0$ if and only if $\varepsilon_{sw} < \hat{\epsilon}$.

i) Assume first that $b > -1$. We derive from the above results that $\hat{\epsilon} > \epsilon_1 > \epsilon_0 > \epsilon_3 > \epsilon_2$. It follows that for any $\varepsilon_{sw} > 0$ there exists at least one negative characteristic root. Moreover, when $\varepsilon_{sw}$ crosses $\epsilon_2$, $\mathcal{P}(-1)$ crosses 0.

ii) Assume now that $b < -1$. We derive from the above results that $\hat{\epsilon} > \epsilon_1 > \epsilon_0, \epsilon_2$ and $\epsilon_0, \epsilon_2 > \epsilon_3$. Again, for any $\varepsilon_{sw} > 0$ there exists at least one negative characteristic root. Moreover, when $\varepsilon_{sw}$ crosses $\epsilon_2$, $\mathcal{P}(-1)$ crosses 0.
7.7 Proof of Proposition 4

Consider the expressions of \( P(1) \) and \( P(-1) \) in the proof of Proposition 2 with \( \varepsilon_{sR} = 0 \) and the difference \( \varepsilon_1 - \varepsilon_2 \) given in (15). Local stability requires \( P(1)P(-1) < 0 \). We easily get the following results:

i) When \( b > 0 \) then \( P(0) > 0, P(-1) > 0 \) and \( P(1) < 0 \) if and only if \( \varepsilon_{sw} < \varepsilon_1 \). Therefore, if \( \varepsilon_{sw} < \varepsilon_1, \lambda \in (0,1) \).

ii) When \( b > 0 \), if there exists a unique steady state \( k^* \), then the proof of Corollary 1 applies so that \( P(1) < 0 \) and \( \lambda \in (0,1) \).

iii) When \( b > -1 \) then \( P(0) > 0 \) iff \( \varepsilon_{sw} < \varepsilon_0 \), \( P(1) < 0 \) iff \( \varepsilon_{sw} > \varepsilon_1 \), and \( P(-1) < 0 \) iff \( \varepsilon_{sw} > \varepsilon_2 \). Moreover, we get \( \varepsilon_1 > \varepsilon_0 > \varepsilon_2 \) and local stability is obtained iff \( \varepsilon_{sw} \in (\varepsilon_2, \varepsilon_1) \). More precisely, we have \( \lambda \in (-1,0) \) when \( \varepsilon_{sw} \in (\varepsilon_2, \varepsilon_0), \lambda \in (0,1) \) when \( \varepsilon_{sw} \in (\varepsilon_0, \varepsilon_1) \), and \( \lambda \) crosses \(-1 \) when \( \varepsilon_{sw} \) crosses \( \varepsilon_2 \).

iv) When \( b < -1 \) then \( P(0) > 0 \) iff \( \varepsilon_{sw} < \varepsilon_0 \), \( P(1) < 0 \) iff \( \varepsilon_{sw} > \varepsilon_1 \), and \( P(-1) < 0 \) iff \( \varepsilon_{sw} < \varepsilon_2 \). Moreover, we get \( \varepsilon_2 > \varepsilon_1 > \varepsilon_0 \) and local stability is obtained iff \( \varepsilon_{sw} \in (0, \varepsilon_1) \cup (\varepsilon_2, +\infty) \). More precisely, we have \( \lambda \in (-1,0) \) when \( \varepsilon_{sw} \in (0, \varepsilon_0) \cup (\varepsilon_2, +\infty), \lambda \in (0,1) \) when \( \varepsilon_{sw} \in (\varepsilon_0, \varepsilon_1) \), and \( \lambda \) crosses \(-1 \) when \( \varepsilon_{sw} \) crosses \( \varepsilon_2 \). \( \square \)

7.8 Proof of Proposition 5

From the characteristic polynomial, since \( \varepsilon_{sR} < 0 \), we have \( \lim_{\lambda \to \pm \infty} P(\lambda) = -\infty \), and local indeterminacy of equilibria requires \( P(1) < 0, P(-1) < 0 \) and \( D < 1 \). Consider also the expression of the discriminant in the proof of Lemma 1. We get \( \Delta \geq 0 \) if and only if

\[
\varepsilon_{sw}\varepsilon_{wr} \frac{b}{1-b} \geq 2\sqrt{-\varepsilon_{sR}\varepsilon_{rk}} + \varepsilon_{sR}\varepsilon_{rk} - 1 - \varepsilon_{py}(1 + \varepsilon_{sR})
\]

We then have

\[
\Delta \geq 0 \iff \varepsilon_{sw} \geq \tilde{\epsilon}
\]

It is also easy to compute the following

\[
\begin{align*}
\tilde{\epsilon} - \varepsilon_1 &= -\frac{1}{b\varepsilon_{sw}}[(1-b)\sqrt{-\varepsilon_{sR}\varepsilon_{rk}} - 1]^2 \\
\tilde{\epsilon} - \varepsilon_2 &= -\frac{1-b}{b(1+b)\varepsilon_{sw}}[(1+b)\sqrt{-\varepsilon_{sR}\varepsilon_{rk}} - 1]^2 \\
\tilde{\epsilon} - \varepsilon_3 &= -\frac{1-b}{b\varepsilon_{sw}}[(1-b^2)(\sqrt{-\varepsilon_{sR}\varepsilon_{rk}})^2 - 2\sqrt{-\varepsilon_{sR}\varepsilon_{rk}} + 1] \\
\end{align*}
\]

1) Since \( b > 0 \) we get \( \tilde{\epsilon} - \varepsilon_1 \leq 0, \tilde{\epsilon} - \varepsilon_2 \leq 0 \). We derive from the expressions of \( P(1), P(-1) \) and \( D \) in the proof of Proposition 2

\[
\begin{align*}
P(1) < 0 &\iff \varepsilon_{sw} < \varepsilon_1 \\
P(-1) < 0 &\iff \varepsilon_{sw} < \varepsilon_2 \\
D < 1 &\iff \varepsilon_{sw} > \varepsilon_3 \\
\end{align*}
\]
Local indeterminacy therefore requires $\epsilon_3 < \epsilon_1$ and $\epsilon_3 < \epsilon_2$. We easily derive from (15) that both inequalities are satisfied if

$$1 + (1 + b)^2 \epsilon_s R \epsilon_r k > 0$$

This condition also implies $\epsilon_1 > \epsilon_2$. Therefore, under the above condition, a given steady state $k^*$ is locally indeterminate if and only if $\epsilon_{sw} \in (\epsilon_3, \epsilon_2)$. On the contrary, saddle-point stability holds when $P(1)P(-1) < 0$ i.e. $\epsilon_{sw} \in (\epsilon_2, \epsilon_1)$.

2) When $1 + (1 + b)^2 \epsilon_s R \epsilon_r k < 0$, we get $\epsilon_3 > \epsilon_2$ and any steady state $k^*$ is necessarily locally determinate. We then have to discuss two cases depending on the respective values of $\epsilon_1$ and $\epsilon_2$.

i) If $1 + (1 + b)(1 - b)\epsilon_s R \epsilon_r k > 0$, we get $\epsilon_1 > \epsilon_2$ and a given $k^*$ will be saddle-point stable if and only if $\epsilon_{sw} \in (\epsilon_2, \epsilon_1)$.

ii) On the contrary, if $1 + (1 + b)(1 - b)\epsilon_s R \epsilon_r k < 0$, we get $\epsilon_2 > \epsilon_1$ and a given $k^*$ will be saddle-point stable if and only if $\epsilon_{sw} \in (\epsilon_1, \epsilon_2)$.

7.9 Proof of Proposition 6

1) We have proved in Corollary 1 that if there exists a unique steady state $\kappa^*$ then $P(1)/b < 0$. When $b < 0$ this implies $P(1) > 0$ and local indeterminacy is ruled out.

2) Recall that a given steady state $k^*$ is saddle-point stable if and only if $P(1)P(-1) < 0$. When $b < 0$, we have

$$P(1) < 0 \iff \epsilon_{sw} > \epsilon_1$$

$$P(-1) < 0 \iff \epsilon_{sw} > \epsilon_2 \text{ if } b > -1 \text{ or } \epsilon_{sw} < \epsilon_2 \text{ if } b < -1 \quad (18)$$

i) Assume first $b > -1$. Then any $k^*$ is saddle-point stable if and only if $\epsilon_{sw} \in (\min\{\epsilon_1, \epsilon_2\}, \max\{\epsilon_1, \epsilon_2\})$.

ii) Assume now $b < -1$ which implies $\epsilon_1 - \epsilon_2 < 0$. Then any $k^*$ is saddle-point stable if and only if $P(1)P(-1) < 0$, i.e. $\epsilon_{sw} \in (0, \epsilon_1) \cup (\epsilon_2, +\infty)$.

7.10 Proof of Proposition 7

Recall that since $\epsilon_s R < 0$, local indeterminacy requires $P(1) < 0$, $P(-1) < 0$ and $D < 1$. When $b < 0$, consider (18) and the following

$$\Delta \geq 0 \iff \epsilon_{sw} \leq \tilde{\epsilon}$$

$$D < 1 \iff \epsilon_{sw} < \epsilon_3$$

Since $b < 0$ we get from (17) $\tilde{\epsilon} - \epsilon_1 \geq 0$.

1) We first study the sub-case $b > -1$ which implies $\tilde{\epsilon} - \epsilon_2 \geq 0$. Local
indeterminacy therefore requires $\epsilon_3 > \epsilon_1$ and $\epsilon_3 > \epsilon_2$. We easily derive from (15) that both inequalities are satisfied if
\[
1 + (1 + b)^2 \varepsilon_{sR} \varepsilon_{rk} > 0 \quad \text{and} \quad 1 + (1 - b)^2 \varepsilon_{sR} \varepsilon_{rk} < 0
\]
Moreover, we get $\tilde{\epsilon} > \epsilon_1$ and $\tilde{\epsilon} > \epsilon_2$. Under these conditions, a given steady state $k^*$ is locally indeterminate if and only if $\varepsilon_{sw} \in (\max\{\epsilon_1, \epsilon_2\}, \epsilon_3)$.

2) Consider finally the sub-case $b < -1$ which implies $\tilde{\epsilon} - \epsilon_2 \leq 0$ and $\epsilon_1 - \epsilon_2 < 0$. Local indeterminacy therefore requires $\epsilon_3 > \epsilon_1$. We easily derive from (15) that this inequality is satisfied if
\[
1 + (1 - b)^2 \varepsilon_{sR} \varepsilon_{rk} < 0
\]
Under this condition, a given steady state $k^*$ is locally indeterminate if and only if $\varepsilon_{sw} \in (\epsilon_1, \min\{\epsilon_2, \epsilon_3\})$.

\[\Box\]

7.11 Proof of Corollary 3

From (17), consider the following polynomial associated with the difference $\tilde{\epsilon} - \epsilon_3$:
\[
(1 - b^2) \left( \sqrt{-\varepsilon_{sR} \varepsilon_{rk}} \right)^2 - 2 \sqrt{-\varepsilon_{sR} \varepsilon_{rk}} + 1 = 0
\] (19)
The two roots are real and are equal to $1/(1 - b)$ and $1/(1 + b)$.

1) Consider first the case $b > 0$. We get from (17) and (19) that $\tilde{\epsilon} > \epsilon_3$ if and only if
\[
\frac{1}{(1-b)^2 \varepsilon_{rk}} > -\varepsilon_{sR} > \frac{1}{(1+b)^2 \varepsilon_{rk}}.
\]
But we know from Proposition 7 that a steady state $k^*$ is locally indeterminate if $-\varepsilon_{sR} < 1/[(1+b)^2 \varepsilon_{rk}]$ and $\varepsilon_{sw} \in (\epsilon_3, \epsilon_2)$. Therefore, the characteristic roots are real for any $\varepsilon_{sw} \in (\epsilon_3, \epsilon_2)$.

2) Consider now the case $b < 0$. Assume first that $0 > b > -1$. We get from (17) and (19) that $\tilde{\epsilon} < \epsilon_3$ if and only if
\[
\frac{1}{(1+b)^2 \varepsilon_{rk}} > -\varepsilon_{sR} > \frac{1}{(1-b)^2 \varepsilon_{rk}}.
\]
Proposition 7 then shows that under this condition, a steady state $k^*$ is locally indeterminate if $\varepsilon_{sw} \in (\max\{\epsilon_1, \epsilon_2\}, \epsilon_3)$. Since $\tilde{\epsilon} - \epsilon_1 \geq 0$ and $\tilde{\epsilon} - \epsilon_2 \geq 0$, we conclude that local indeterminacy with complex characteristic roots occur when $\varepsilon_{sw} \in (\tilde{\epsilon}, \epsilon_3)$.

Assume finally that $b < -1$. We get from (17) and (19) that $\tilde{\epsilon} < \epsilon_3$ if and only if
\[
-\varepsilon_{sR} > \frac{1}{(1-b)^2 \varepsilon_{rk}},
\]
Proposition 7 then shows that under this condition, a steady state $k^*$ is locally indeterminate if $\varepsilon_{sw} \in (\epsilon_1, \min\{\epsilon_2, \epsilon_3\})$. If moreover $\epsilon_2 > \epsilon_3$, i.e.
\[
\frac{1}{(1+b)^2 \varepsilon_{rk}} > -\varepsilon_{sR},
\]
we conclude that local indeterminacy with complex characteristic roots occur when $\varepsilon_{sw} \in (\tilde{\epsilon}, \epsilon_3)$.

\[\Box\]
References


