On Some Macroscopic Origins and Consequences of Economic Inequality: An Evolutionary Perspective*

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Abstract

Income inequality can vary rather dramatically across societies. While in some countries the average income of the richest 10% does not exceed 5 or 6 times that of the poorest 10%, in others the same ratio can reach up to 90 or 100. Moreover, such differences can persist and even increase over long periods of time. In order to address such facts, we develop a theory, based on socio-cultural evolution, that highlights the stability and heterogeneity of a society’s economic environment as a fundamental source of long-term inequality. We show that steady and diverse economic environments provide a selective advantage for the evolution of equality, whereas fluctuating and singular ones promote inequality. We also show that more equal societies exhibit a higher degree of social mobility and are more resilient and robust in the sense of being quicker to recover from shocks and to return to normalcy than unequal ones. We thus provide a rationale for the emergence of inequality, its persistence and negative correlation with social mobility, as well as its role in determining the fragility of a society.

Keywords: Income inequality; social mobility; macroeconomic volatility; cultural evolution; evolutionary entropy. JEL Classification: E24, E32, O11, O13, O15

1 Introduction

The question of why some societies are wealthier and/or exhibit stronger growth than others has received much attention within economics, as has the question of possible consequences of income inequality for development and growth (Acemoglu [1], Aghion et al. [3], Bénabou [10], Galor [28], Ray [41]; see also Lucas [36] for a reduced form model). Somewhat less studied is the question of why societies differ so much in terms of inequality; and why such different levels of inequality persist for so long across societies (Acemoglu and Robinson [2], Atkinson [7], Atkinson and Bourguignon [8], Boix [11], Piketty [39]). We here introduce an empirically grounded, evolutionary framework that provides an explanation for why some societies are

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stratified, economically fragile, and characterized by inertia of social mobility, while others are highly egalitarian, resilient towards shocks, and exhibit high levels of intergenerational mobility. An advantage of our approach is that it allows, within a single framework, to account for three critical dimensions of the problem, namely, the spread of equality/inequality, the correlation between income inequality and social mobility across generations, and the social and economic instability of highly stratified societies. To the best of our knowledge this has not been addressed so far in a single or unified framework in the economics literature.

Inequality has traditionally been addressed in economics in terms of supply and demand for different types of factor endowments (Atkinson and Bourguignon [8] contains a survey). Within this class of models, economic inequality emerges from the differences of remuneration for the different factors. These models are typically formulated within the context of competitive markets and are sufficiently flexible to provide insight into a variety of empirical aspects of inequality, including the explanation of a widening gap between skilled and unskilled labor due to technological change. An important limitation of these models, however, is that they take as given the distribution of factor endowments, as well as the pricing mechanism that ultimately establishes the compensation of the different factors. Not surprisingly, they are unable to explain two empirical observations that set apart stratified and egalitarian societies, namely, (i) the correlation between economic inequality and intergenerational mobility: intergenerational earning mobility is low in countries with high income inequality such as several countries in Latin America, Africa, and the Middle East, and is high in Scandinavian countries where income distribution is less stratified and more egalitarian; (ii) the correlation between economic inequality and economic (and political) stability or resilience, that is, the capacity of a society and its economic institutions to maintain a functional economic (and political) network in spite of internal and external shocks; resilience is characteristic of egalitarian societies such as the Nordic countries; the social and economic fragility of stratified societies is strikingly illustrated by the recent social and economic disruptions observed in Libya, Tunisia, Egypt, and Syria.

More recently, the economics literature on inequality has also addressed questions of persistent inequality and intergenerational mobility (Piketty [39] contains a survey). These have helped to elucidate the shortcomings of the supply and demand model and have delineated several socio-cultural factors which contribute to the origin and persistence of inequality. These include: (i) the transmission of wealth from parents to children through inheritance, which can amplify economic inequality and facilitate its persistence across generations; (ii) the intergenerational transmission of ambition and the recognition of economic success and social prestige; underlying this is the hypothesis that individuals tend to compare their social achievements to the reference groups to which they belong, such that individuals with lower class origin are less motivated than individuals with upper class origins to make human capital investments that may enhance their social status; (iii) statistical discrimination or the prevalence of self-fulfilling discriminatory beliefs, whereby persistent generational inequality between two social groups with homogeneous characteristics can be generated by discriminatory hiring policies based on certain assumptions regarding the abilities of the individuals that belong to the different groups.

This socio-cultural perspective resolves certain anomalies of the supply and demand models. But it also has its shortcomings in that it does not account for the fact that these features, the transmission of wealth through inheritance, the intergenerational transmission of ambition and work ethic, and the extent of discrimination, also show a large variation across societies.\(^1\)

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\(^1\)Empirical support for differences in the intensity of statistical discrimination between societies is starkly illustrated in a recent study of anti-semitism in Germany over a period of 600 years by Voigtländer and Voth [49]. These authors show that discrimination against Jews was highly correlated with the economic environments; in particular, discrimination was lower
Societies which show significant differences in economic inequality typically differ in terms of their macroeconomic environment as defined in terms of steadiness and diversity of its resources. Countries with severe economic inequality such as Libya, Saudi-Arabia, or Nigeria, have economies which are dependent on a singular resource, namely oil, whose economic benefits are often non-constant. Countries with little economic inequality, such as the Scandinavian countries have economies based on more diverse resources with relatively more constant benefits. This is consistent with the observation that the macroeconomic environment may be a critical factor in generating persistent inequality.

The model we develop accounts for this by modeling certain characteristics of the economic environment as basic driving forces implicated in the redistribution of resources in society. The idea is that modes of behavior, cooperative or self-serving, are correlated with more or less egalitarian income redistribution, and that the overall macro-economic environment confers certain advantages to different modes of behavior. Intuitively speaking, cooperative behavior will have a selective advantage in societies where the economic environment is steady and diverse; self-serving behavior will have a selective advantage in societies with environments that are variable and singular. Accordingly, steady and diverse economic environments and variable and singular ones, will result in different modes of behavior that will result in lower or higher income inequality.

These two types of environments – steady and diverse versus variable and singular – are the extremal states which we distinguish in the analytical formulation of our model. Our heuristic explanation of the large variation in income inequality from highly stratified to highly egalitarian societies is based on an evolutionary analysis of the spread of cooperation, in different environments, by means of three main forces: selection, which discriminates between behavioral dispositions according to the economic benefit the behavior confers to the individuals that adopt it; inheritance, a cultural mechanism which specifies how behavioral traits are transmitted between individuals that share certain social or phenotypic characteristics; variation, individuals in a population vary in terms of their behavioral traits; these differences are largely the derivations of different socio-economic conditions under which the individuals are raised.

At a more formal level, our analysis revolves around a statistical measure of interaction called evolutionary entropy. This quantity describes the extent to which individual groups in a society share and allocate resources between each other. It measures the number and intensity of distinct pathways of income or commodity flows between individuals in the network. A low entropy social network is described by a poverty of pathways of flows. Such a network is characterized by limited interactions between groups or by a single group of individuals with more pathways being directed towards it. A high entropy network is described by a diversity of interactions and by a large number of pathways between individuals in the different groups.

The significance of the evolutionary entropy concept towards our understanding of the origin of stratified or egalitarian societies resides in its relation to certain key variables. In the paper, we formally derive relationships between our entropy measure and the following variables:

- **Inequality**: Evolutionary entropy is negatively correlated with the Theil index, which is a standard, entropy-based measure of income inequality. This means that societies with high entropy will tend to have low income inequality, whereas societies with low entropy will have high income inequality; and less persistent in regions with high levels of trade, (which we expect to be both more constant and diverse). In particular, it is consistent with the idea that the external economic environment is a relevant force for the emergence of dispositions, cooperative or self-serving, that also determine the level of discriminatory behavior and the spread of economic inequality. Notice also, that the different levels of concentration of landownership in Prussia documented by Ziblatt [51, 52] are also consistent with our approach.
- **Social Mobility**: It is positively correlated with statistical measures of income mobility and of social mobility. Thus societies with low economic inequality will exhibit high income and social mobility, whereas societies with large economic inequality will exhibit low mobility; and finally

- **Resilience**: It is positively correlated with the stability or resilience of the social network. Thus economic inequality also relates to the resilience of the society to withstand shocks. Egalitarian societies are highly resilient and resistant to shocks and to disruptions of their basic structure; stratified societies are highly sensitive to such structural perturbations.

The concept of evolutionary entropy has its origins in the ergodic theory of dynamical systems (Arnold et al. [6], Demetrius [19, 20]). This mathematical object is the cornerstone of directionality theory (Demetrius [20]), an analytic model of evolutionary forces of variation and natural selection. One of the central tenets of the theory is the **entropic selection principle**. This principle states that the dynamical changes in evolutionary entropy under the process of variation and selection are contingent on the resource abundance and diversity. These changes can be characterized in terms of the following tenets: when resources are constant in abundance and diverse in composition, communities with higher evolutionary entropy will have a selective advantage and increase in frequency; when resources are fluctuating in abundance and are singular in composition, communities with low entropy will have a selective advantage and will increase in frequency. The analytical derivation of this principle is based on the integration of the ergodic theory of dynamical systems with the theory of diffusions processes.

We exploit this to shed light on three key socio-economic phenomena, namely, the large variation in economic inequality across countries, social mobility and the robustness of societies, which have hitherto seemed untractable within the economics literature on the origin and spread of inequality. The main results we obtain can be qualitatively described as follows.

**(1) Redistributive Selection Theorem.** When the income generation process is steady and based on a diverse set of activities and resources, a cooperative disposition will spread and communities will tend to a condition whereby economic and social redistribution are equal. When the income generation process is fluctuating and based on a narrow set of resources and activities, selfish disposition will have a selective advantage and societies will evolve towards states where economic and social redistribution are unequal.

This provides a new perspective on Adam Smith’s invisible hand in relation to inequality. According to Adam Smith, the private pursuit of self-interest would lead always, as if by an invisible hand, to the well-being of all. Our analysis indicates that this condition of well-being of all will only be achieved in the long run if the environment and external resource conditions are constant and diverse. If the environment is fluctuating and singular beyond a certain threshold, then the invisible hand would lead in the long run to a situation where the well-being of a few is achieved.

**(2) Social Mobility Theorem.** Societies with a steady and diverse income generation process will be characterized by high intergenerational mobility; whereas societies with a fluctuating and singular income generating process will be characterized by low social mobility.

This constraint on social mobility further increases the degree of inequality in the society and drives society towards greater instability and inefficiency. Thus societies with important income inequality become increasingly unequal, inflexible, and stagnant. By putting the steadiness of the environment at the center
of the analysis the theory provides a new perspective on the long-standing debate in sociology of whether industrialist-capitalist societies tend towards more social mobility or whether they tend towards less social mobility, that is whether they converge to stratified capitalist dynasties (Piketty [39]). From the viewpoint of our theory, the debate has possibly overlooked an important explanatory variable, namely, the steadiness and diversity of the economic environment. Finally, the instability of unequal societies is reflected in our third main result.

(3) Perturbation Stability Theorem. Societies described by unequal distribution of resources – highly stratified societies – are inherently unstable and highly sensitive to small perturbations in resource allocation. They take longer to return to normalcy than more equal ones.

This captures part of the large inefficiency and instability observed in highly stratified countries, e.g., the recent events in the Middle East (specifically in Egypt, Tunisia, Libya, Syria among others). These can be seen as instances of this theorem, where relatively small shocks can lead to major disruptions affecting the everyday life and organizational structure of an entire country for a prolonged period of time.

Further Related Literature. Within the vast literature on macroeconomics, growth, and inequality a number of papers have studied the link between inequality and uncertainty or macroeconomic volatility. Some of these papers link inequality to agents’ uncertain endowments and different attitudes towards risk (Caroli and García-Peñalosa [13], Checchi and García-Peñalosa [14]), or through saving and the labor supply (García-Peñalosa and Turnovsky [29]), or directly through various direct or indirect effects of the business cycle that amplify the inequalities (Stiglitz [46, 47]). Some papers study the link empirically (Breen and García-Peñalosa [12], Laursen and Mahajan [34], Huang et al. [31]) and find a positive correlation between income inequality and macroeconomic volatility, measured as the standard deviation of the rate of growth of real per capita GDP, which we will come back to in Section 7.

The paper is organized as follows. Section 2 presents the main framework adapted from directionality theory. Here we articulate the basic framework which we use to address the problem of the origin and evolution of income inequality. In Section 3, we present an example of a simple economy to illustrate some of the main variables introduced. Section 4 contains the main analytical result, the entropic selection principle. Section 5 establishes the relation between evolutionary entropy and various measures of inequality such as the Theil index as well as a standard measure of income mobility and another measure of social mobility, and thus states the main economic results of the paper. Section 6 addresses the instability of unequal societies and discusses some possible consequences. Section 7 briefly discusses some empirical evidence from the existing literature, and Section 8 concludes with a few policy implications of the theory. All the proofs and several basic concepts from directionality theory are contained in the Appendix.

2 The Framework

Our analysis is based on a model of the cultural evolution of societies. The basic building block are individuals that are organized in groups or classes and interact with each other in the production and allocation of resources. Thus we distinguish two main processes, one describing the evolution of the population ($N(t)$) and another one describing the evolution of aggregate income or production ($Y(t)$) in the economy. The interaction between the individuals in the society is at the center of our theory and is captured by what we call an interaction matrix $A$ described below.
Population and Income. Consider a society with a total population $N(t)$ of individuals distributed in $d$ classes, written as,

$$N(t) = \sum_{i=1}^{d} n_i(t),$$  \hspace{1cm} (1)

where $n_i(t)$ denotes the number of individuals (or households) in class $i$ in period $t$. The classes may be thought of as describing occupational classes, which we assume throughout to be of equal size $n_i(t) = N(t)/d$. These individuals engage in several activities in order produce and exchange commodities and services. The total income (or production) $Y(t)$ of the economy,

$$Y(t) = \sum_{i=1}^{d} y_i(t),$$  \hspace{1cm} (2)

is the sum of income going to the different classes, where $y_i(t)$ denotes the amount income going to class $i$ in period $t$; we also write $y(t) = (y_i(t))$ for the vector of incomes of the different classes. Prices are not modeled explicitly in this set-up. Implicitly we assume all goods (outputs, inputs, services) are exchanged against a numeraire commodity and it is the transfers of this numeraire commodity during the given periods of time that we record here. As is standard with population models, the discreteness and finiteness of the income and population processes is an important element in the present analysis (e.g., Dawson [18] and Haccou et al. [30]).

Interaction Matrix, Production, and Allocation. Besides trading and transforming resources for themselves the agents also transfer income (and goods) to each other, including across classes. The way in which income is created and transferred across agents is a fundamental characteristic of the society that as we will see plays a determining role for its long-run development. We capture this process compactly by means of the interaction matrix,

$$A = (a_{ij}), \hspace{0.5cm} a_{ij} \geq 0, 1 \leq i,j \leq d,$$  \hspace{1cm} (3)

where $a_{ij}$ measures the marginal rate of “contribution” of a unit of income in class $j$ in period $t$ towards income in class $i$ in period $t+1$. The “contributions” can derive from income generation or from the exchange of commodities or services. We assume that all transactions are made in discrete units – “representative” units of the numeraire good – and that the (finite) number of units generated are random but occur according to the fixed rates $a_{ij}$, which constitute per period averages.

The interaction matrix $A$ is at the center of our analysis It describes both production and allocation of commodities held by individuals in the society. At the same time, it defines the steady state law of motion

\footnote{All results go through if this only holds to an approximate degree, $n_i(t) \approx N(t)/d$. We do not elaborate on this.}
of aggregate income,\(^3\)

\[ y(t + 1) = Ay(t). \]  

(4)

It can be thought of as representing a directed graph over \(d\) nodes (the \(d\) classes or occupations), where \(a_{ij} > 0\) corresponds to a directed link from node \(j\) to node \(i\) of intensity \(a_{ij}\). The structure of the underlying graph reflects the social preferences of the individuals. For simplicity, we assume the matrix is irreducible, meaning that all matrix entries are strictly positive, implying the underlying directed graph is strongly connected; for any pair of nodes \(i\) and \(j\) there is a directed path going from \(i\) to \(j\) and from \(j\) to \(i\) of strictly positive intensity.

**Steady State Dynamics.** In general one can imagine the economy following a non-linear dynamics of the form,

\[ y(t + 1) = A(t)y(t), \]

where the matrix \(A(t) = (a_{ij}(t))\) is more generally a matrix varying with time and whose entries can also depend on the distribution \(y(t)\), where \(y(t) = (y_i(t))\) measures the amount of income of the different classes, \(i = 1, \ldots, d\) at period \(t\). Our focus in this paper is in the economy once it has evolved to the steady state, such that the entries of the interaction matrix are all fixed and constant. The steady state distribution is then obtained by computing the right eigenvector, \(v = (v_1, \ldots, v_d) \in \mathbb{R}^d_+\), corresponding to the dominant (maximal) eigenvalue \(\lambda \in \mathbb{R}_+\), such that

\[ Av = \lambda v. \]

(5)

The magnitude \(\bar{v}_i = v_i / \sum_{j=1}^d v_j\) measures the steady state share of aggregate income or product of the class \(i, i = 1, \ldots, d\). We also refer to \(r = \log \lambda\) as the **growth rate** of the steady state path.\(^4\)

**Evolutionary entropy.** From the interaction matrix \(A\) one can derive the **Markov matrix**

\[ P = (p_{ij}) = \left(\frac{a_{ij}v_j}{\lambda v_i}\right), \quad 0 \leq p_{ij} \leq 1, 1 \leq i, j \leq d. \]

describing the probabilities of transferring “representative” units of goods and income across classes. An element \(p_{ij}(\geq 0)\) can be interpreted as the probability that, in the steady state, a unit of the numeraire good of an individual in class \(i\) originates from an individual in class \(j\).\(^5\) Define \(\pi = (\pi_1, \ldots, \pi_d)\) as the

\(^3\)The balanced growth model of Solow and Samuelson [44] is perhaps the closest to our present approach. They consider a productive environment of a closed economy for which “outputs become inputs one unit of time later” so that they formulate the intertemporal relation between quantities of \(d\) commodities (inputs and outputs) produced in period \(t + 1\) with the quantities of the same commodities produced in period \(t\). This gives rise “casual system of nonlinear difference equations,”

\[ y_i(t + 1) = H_i(y_1(t), \ldots, y_d(t)), \quad i = 1, \ldots, d, \]

or \(y(t + 1) = H(y(t))\) for short. However, if one assumes that the map \(H\) is linear, the system reduces to,

\[ y_i(t + 1) = h_{i1}y_1(t) + \ldots + h_{id}y_d(t), \quad \text{for } i = 1, \ldots, d, \]

or \(y(t + 1) = Hy(t)\) for short. The coefficients \(h_{ij} \in \mathbb{R}_+\) can then be interpreted as marginal productivities of the different commodities produced in period \(t\) in the production of commodity \(i\) produced in period \(t + 1\). Our framework shares with this approach the way of modeling an intertemporal production process. An important difference is that we also incorporate an allocative dimension in the sense that, in our model, \(y_i(t)\) is income held by class \(i\), and what we describe represents the society’s income generation process rather than a pure production process. Our matrix \(A\) describes the intertemporal relationship between different amounts of income distributed across the \(d\) different classes in period \(t\) and the income in the same classes in period \(t + 1\).

\(^4\)Thus, if at time 0 we have income levels \(v(0) = (v_1(0), \ldots, v_d(0)) \in \mathbb{R}^d_+\), then for any time \(t > 0\), we expect, \(E_0[v(t)] = \lambda^tv(0) = (\lambda^tv_1(0), \ldots, \lambda^tv_d(0)) = (e^{\lambda t}v_1(0), \ldots, e^{\lambda t}v_d(0)) = e^{\lambda t}v(0)\).

\(^5\)This Markov matrix \(P\) is central to the derivation of the representation of the process \(Y(t)\) as a diffusion process.
corresponding stationary distribution satisfying $\pi P = \pi$, where an element $\pi_i(\geq 0)$ reflects the probability (in the sense of fraction of time) that a given unit of income is with individuals in class $i$. Then we can define the **evolutionary entropy** of the society as

$$H = -\sum_{i=1}^{d} \pi_i \sum_{j=1}^{d} p_{ij} \log p_{ij}, \quad 0 \leq H \leq \log d,$$

(6)

where $H = 0$ indicates minimal, and $H = \log d$ maximal, entropy. This is the central concept of the present approach.\(^6\)

The concept of evolutionary entropy describes the number and intensity of pathways of commodity flows between the individuals in the social network. It is thus a measure of the strength of the interactions of the productive and allocative process between the individuals in society.\(^7\) The higher the number of links or flows between agents within and across classes the larger the entropy. As we will see in Theorem 4, it is also a measure of stability of the society in that it is a measure of the rate at which the society returns to the steady state after a shock.

Egalitarian and stratified societies are represented in graphs of Figure 1. In the left-hand graph there are many distinct pathways of resource flows, thus describing an egalitarian or high entropy interaction network. In the right-hand one the pathways are essentially directed towards a single group, (here $a \gg \epsilon$), thus describing a stratified or low entropy network.

**Fluctuations around the Steady State.** Given the steady state law of motion described by (4) and the discreteness and finiteness of the units transacted, the actual aggregate production $Y(t)$ fluctuates around the steady state.\(^8\) In the Appendix we sketch the underlying probabilistic structure derived from the above

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\(^6\)The notion of evolutionary entropy used in this paper is to be contrasted with the notion of entropy used in thermodynamics. Boltzmann and Gibbs defined a statistical notion of entropy $S = -\Sigma j p_j \log p_j$, where $p_j$ is the probability that a randomly chosen particle is in energy state $j$, while Clausius characterized it in energetic terms, $dS = \frac{dQ}{T}$, where $Q$ is the heat of the material body and $T$ is its temperature. The thermodynamic notion of entropy applies to isolated systems and hence to inert matter. It also measures the extent of energy spreading and sharing, and is thus generalized by the notion of evolutionary entropy which applies to living organisms and biological systems. The evolutionary entropy notion also has an energetic characterization besides the statistical one given in this paper. For more discussion, see Demetrius [20].

\(^7\)Although this is not formally derived, in the introduction we loosely interpret the evolutionary entropy as being associated with the degree of cooperativeness of the society. The idea is that a strong degree of interaction measured by the number of pathways between individuals in different classes appears to be related to high level of cooperation.

\(^8\)This is a result that is well studied in stochastic population dynamics following the branching processes originally analysed by Feller [25] (see Dawson [18] and Haccou et al. [30] for more recent expositions). In these models there is a natural unit, an individual in the population; in the present approach, we assume the existence of a “representative” unit of production and trade that is also indivisible. Our model records the production and transactions of such units.
Markov matrix $P$, under which $Y(t)$ can be represented by a process which can be described as the solution to the following continuous time stochastic differential equation,

$$dY(t) = rY(t)dt + \sigma \sqrt{Y(t)}dW(t),$$

(7)

where $W(t)$ is Brownian motion, and $r \in \mathbb{R}$ and $\sigma \sqrt{Y(t)} \geq 0$ describe respectively the growth rate and the standard deviation of the process; the macroscopic variable $\sigma$ is defined in the next subsection.\(^9\) Essentially, the process $Y(t)$ can be viewed as a branching process (of the Galton-Watson type), where in any given interval of time different units of the “representative” commodities are generated according to rates derived from the matrix $A$. This gives rise to the Feller type process described in (7) that ultimately only depends on two parameters, $r$ and $\sigma^2$ (see Dawson [18] and Haccou et al. [30]).

**Macroscopic Variables.** The evolution of aggregate income is subject to variations, perturbations, and shocks, that can originate in a variety of ways, reflecting different types of innovations, technological, organizational, political, or cultural, but also events such as small-scale conflicts, epidemics, immigration among many others. Indirectly the interaction matrix $A$ embodies the degree of efficiency in appropriating, transforming, and allocating resources, commodities, or income. In many cases these variations (perturbations, innovations, or shocks) affect the production process and the way the society interacts; in our analysis they are captured by small changes in the interaction matrix. It is important therefore to characterize how the evolution of aggregate income responds to such small perturbations of the interaction matrix. In order to keep things tractable while attempting to capture a large class of such variations, we consider, throughout the paper, perturbations of the form,

$$A(\delta) = (a_{ij}^{1+\delta})_{ij} = A^{1+\delta}, \text{ for } \delta \in \mathbb{R}. \quad (8)$$

These constitute in a precise sense a canonical class of perturbations that can be modeled with a one-dimensional parameter.\(^10\) These perturbations play a fundamental role in the underlying analysis of our approach (comparable to mutations in biological models). Importantly, they generate a number of **macroscopic parameters** that measure diverse aspects of the income process $Y(t)$ besides its growth rate which has been almost exclusively at the center of most economic growth theory (e.g., Acemoglu [1], Aghion and Howitt [4], and Galor [28]). In our analysis, the growth rate $r$ and the evolutionary entropy $H$ are the two most important ones, but as we will see there are several others that also play a vital role.

We capture the effect of a perturbation on the steady state growth path by means of a Taylor expansion of the growth rate of aggregate production,

$$r(\delta) = r(0) + \delta r'(0) + \frac{\delta^2}{2!} r''(0) + \frac{\delta^3}{3!} r'''(0) + \cdots,$$

(9)

where $r(\delta) = \log \lambda(\delta)$ is the perturbed growth rate. From this we derive various **macroscopic parameters**

\(^9\)This process is likely to underestimate the actual variance, see Stock [48] who estimates a variance of the order $Y(t)$ rather than $\sqrt{Y(t)}$ for postwar US GNP. To better understand the difference, write the discrete time version of our process as $Y(t) = (1 + r)Y(t-1) + \epsilon_t$, where $\epsilon_t \sim N(0, \sigma^2 Y(t))$, and notice that our process yields an error term that decreases in $Y(t)$, hence vanishes in the limit, when estimated in log-differences ($\Delta \log Y(t)$) rather than a stationary one as in [48].

\(^10\)We refer to Arnold et al. [6], Demetrius et al. [22], and Demetrius [20], Ch. 4, for a more formal statement of this fact; we provide a sketch of the argument in Appendix A.2. Perturbing individual entries of the matrix $A$ can be both arbitrary and may violate certain feasibility conditions of how resources can be transformed and transferred across the society given its level of technology. The one-parameter family we consider is to be seen as a tractable compromise to capture a wide class of phenomena.
that are useful in characterizing the long-run evolution of the process \( Y(t) \). As we will see, besides the growth rate \( r \) and the evolutionary entropy \( H \), a fundamental parameter is the first derivative of the growth rate, \( \Phi = r'(0) \); other relevant parameters are the second and third derivatives, \( \sigma^2 = r''(0) \) and \( \kappa = r'''(0) \) respectively; another parameter that plays an important role is \( \gamma = \frac{\partial \sigma^2}{\partial \delta} \bigg|_{\delta=0} \), which is the first term of the Taylor expansion of \( \sigma^2(\delta) \).\(^{11}\) An important feature of the present evolutionary approach is that the signs and magnitudes of the first three moments matter in the long-run evolution of aggregate production and its distribution. Consistent with the literature, we sometimes refer to \( \Phi \) and \( \gamma \) as respectively the reproductive potential and the demographic index; as we will see, they are the macroscopic variables representing, respectively, constancy and diversity of the income process (see the next subsection below), and they can be shown to satisfy the relations \( \Phi = r - H \) and \( \gamma = 2\sigma^2 + \kappa \) (see Appendix A.2 for a derivation of this and other relations between the macroscopic variables).

**Constancy and Heterogeneity of the Economy.** The entropic selection principle, as the central tenet of our theory, distinguishes the income generation processes or the productive environment in which the economy operates along two dimensions. On one hand it distinguishes environments by how steadily the income process evolves, and on the other, by how rich and diverse their base of income generation is. We refer to these two aspects as the constancy and heterogeneity dimensions. As is clear especially from the proof of the entropic selection principle, these two aspects are determinant for the direction of the society’s cultural-evolutionary behavior. We define them formally.

**Constancy:** We say the income generation process is **constant** or steady if \( \Phi < 0 \), that is when the growth rate \( r \) is smaller than the evolutionary entropy \( H \), so that the coefficient \( \Phi = r - H \) is negative; we say it is **fluctuating** or unsteady if \( \Phi > 0 \), that is, when the growth rate is larger than the entropy, so that \( \Phi = r - H \) is positive.

As will be clear from the Perturbation Stability Theorem (Theorem 4) the evolutionary entropy \( H \) is a direct measure of the rate at which the economy returns to steady state after any given shock. This suggests that when \( \Phi = r - H > 0 \) the economy grows faster than it returns to steady state, thus resulting in an unsteady or what we call “fluctuating” environment; when \( \Phi < 0 \) the economy grows at a slower rate than the one with which it returns to steady state, thus resulting in a more steady or what we call “constant” environment. In Appendix A.3 we also formally show a positive correlation between \( \Phi \) and the standard economics measure of volatility of the income process \( Y(t) \), namely, the variance of the short term growth rate of aggregate income.

**Heterogeneity:** We say the income generation process is **diverse** if \( \gamma > 0 \), that is when the kurtosis measure \( \kappa \) is nonnegative or when the variance measure \( \sigma^2 \) is large relative to the kurtosis measure \( \kappa \), so that the coefficient \( \gamma = 2\sigma^2 - \kappa \) is positive; we say it is **singular** if \( \gamma < 0 \); this can only occur if the kurtosis measure \( \kappa \) is negative and the variance measure \( \sigma^2 \) relatively small.

Given the nonnegativity of the variance, a small or negative value of \( \gamma \) essentially reflects a negatively skewed income process. This is typical of economies with few important activities rather than many relatively not so dominating ones.

\(^{11}\)By this we mean the Taylor expansion associated to \( \sigma^2(\delta') = r''(0)(\delta') \) when further perturbing the perturbed matrix \( A(\delta) \) again by \( \delta' \), giving the matrix \( A(\delta')(\delta') \).

\(^{12}\)Our framework does not directly distinguish different “types” of activities. As such we cannot give a formal derivation of the connection between our measure \( \gamma \) and an empirically observable notion of diversity. However, as is clear from Appendix A.3, the variable \( \sigma^2 \) is a measure of variance derived from the \( a_{ij} \) terms. A large number of activities will likely be associated with a large such variance and will result in a positive value of \( \gamma = 2\sigma^2 + \kappa \). On the other hand, a small number of activities is likely to be associated with a small variance and, moreover, if the productivities associated are also concentrated in few highly performing activities, this is likely to generate a distribution of the \( a_{ij} \)’s which is negatively skewed (high density on
Population. As we will discuss in more detail below, we also assume that total population \( N(t) \) follows a stochastic process described by the solution to the stochastic differential equation

\[
dN(t) = \bar{r}N(t)dt + \sigma \sqrt{N(t)}dW(t),
\]

(10)

where again \( W(t) \) is Brownian motion in common with the income process \( Y(t) \). Thus to capture the fact that individual reproduction is positively related to the income of the individuals, we assume that in the steady state the population process \( N(t) \) is coupled with the production process \( Y(t) \) by means of a linear relation \( (F,f) \) such that \( N(t) = F(Y(t)) \) and \( n(t) = f(y(t)) \), where \( F', f' > 0 \). It is important to point out that while we assume the processes \( Y(t) \) and \( N(t) \) are positively and linearly related, what really matters for our results is not so much the linearity but rather a strictly positive relation between the variables that has second order derivatives bounded from below. This ensures that key macroscopic variables such as the growth rates and variances of the two processes are strictly positively correlated (as specified in Appendix A.3). We will describe the population process in more detail in Section 4 below. We assume linearity of the relation \( (F,f) \) for simplicity and in order to be able to characterize the processes \( Y(t) \) and \( N(t) \) as diffusions satisfying respectively (7) and (10) with common process \( W(t) \).

Cultural Evolution and Entropic Selection Principle. Having described the main ingredients of the framework, we can now turn to the model of cultural evolution that we use to study the evolution of inequality in our economies. The model of cultural evolution we consider is based on studying the interaction between an incumbent population and a variant population that can potentially increase in frequency and lead to a displacement of the traits of the original type. This can be seen as being representative of various models of cultural evolution. We then calculate the probability that the invader population takes over the whole population and relate this event to macroscopic parameters of the two underlying income processes involved.

More specifically, to the incumbent population \( N \) operating with an interaction matrix \( A \) as described above, we consider a (small) population of invaders \( N^* \) operating with an interaction matrix \( A^* \) that is a perturbation of the one of the incumbent population, \( A^* = A(\delta) \). The introduction of a variant type constitutes the first event in describing the evolutionary dynamic. The second event is the competition between the incumbent population \( N \) and the variant population \( N^* \) for the resources which the economic environment produces. The evolutionary entropies \( H \) and \( H^* \) associated with \( A \) and \( A^* \) respectively reflect the social preferences and ultimately the interaction mode of the populations \( N \) and \( N^* \). Our model naturally assumes that resources are limited in abundance. This means that the outcome of the competitive process between the two populations will be decided by the respective rates at which they can appropriate resources from the external environment. The selective outcome is then decided by their respective evolutionary entropies \( H \) and \( H^* \) and is contingent on the variability and heterogeneity of the environment.

Formally, the (global) selective dynamic is determined by the entropic selection principle. As studied in directionality theory, the dynamical changes in evolutionary entropy under the process of variation and selection are contingent on the resource abundance and can be characterized in terms of the following local result:

(I) When the income generating process is constant and diverse, communities with higher entropy will have activities with high values of \( a_{ij} \), this will lead to a negative kurtosis \( \kappa \), and will overall result in a negative value of \( \gamma \). We leave a formalization of this relationship for future research.
have a selective advantage and increase in frequency.

(II) When the income generating process is fluctuating and singular, communities with low entropy will have a selective advantage and will increase in frequency.

We will exploit (I) and (II) to resolve a series of problems on the origin, spread, and persistence of inequality, which have hitherto seemed intractable within classical frameworks of economic growth and cultural evolution. The entropic selection principle can also be interpreted loosely as characterizing when cooperative vs. self-serving behavior, or more vs. less egalitarian social preferences, will spread locally.

3 Two Examples

To illustrate the framework and especially to give a more concrete interpretation of the interaction matrix $A$ that is at the center of our analysis, we sketch two examples. The first example, which is an extension of the basic model of Solow and Samuelson [44], considers a multi-sector economy where the classes correspond to the sectors. We describe and interpret the law of motion for such an economy. The second example builds on a one-sector model of Alesina and Rodrik [5], where individuals from two classes of agents work together to produce a single homogeneous commodity. Again, we derive the law of motion for a simple calibration of the model.

Example 1. Following Solow and Samuelson’s [44] model mentioned in Footnote 3 above, consider a closed economy with $d$ sectors such that each sector produces its commodities using as potential inputs, commodities produced by sectors $1, \ldots, d$. Suppose also that individuals are associated to a sector in the sense that they work in just one sector. The population is then evenly distributed across $d$ classes that coincide with the $d$ sectors. Suppose also that individuals in the different classes consume (as outputs) commodities from the same sectors $1, \ldots, d$. Without modeling the details of the individual consumption and production decisions, assume the relative proportions are all linearly additive.\(^{13}\) Then, let $y(t) = (y_1(t), \ldots, y_d(t))$, where $y_i(t)$ is total income of class $i$ in period $t$. We can write down the law of motion for the corresponding economy in the form the following difference equations,

$$ y_i(t + 1) = a_{i1}y_1(t) + \ldots + a_{id}y_d(t), \quad \text{for } i = 1, \ldots, d, $$

or more compactly as,

$$ y(t + 1) = Ay(t), $$

where $A = (a_{ij})$ is an interaction matrix. But more explicitly, since there is a consumption and production dimension in this economy, the elements of the interaction matrix could be written more explicitly as the sum of a production and a consumption component, that is, for each $i = 1, \ldots, d$, we can write,

$$ y_i(t + 1) = a_{i1}y_1(t) + \ldots + a_{id}y_d(t) = (a_{i1}^c + a_{i1}^p)y_1(t) + \ldots + (a_{id}^c + a_{id}^p)y_d(t), $$

where

$$ a_{ij} = a_{ij}^c + a_{ij}^p > 0. $$

\(^{13}\)This can be made consistent with a simplified multi-sector growth model along the lines of, for example, Acemoglu [1], Ch. 20. However, while the present example provides a clear interpretation of the matrix $A$, the next example with one sector and two classes seems more representative of modern industrialized economies, where individuals from different classes can work in the same sector.
and where \( a_{ij}^c \geq 0 \) represents the marginal contribution of a unit of income in class \( j \) at time \( t \) to a unit of income in class \( i \) at time \( t + 1 \) due to the consumption part of individuals’ activity in class \( j \); similarly \( a_{ij}^p \geq 0 \) captures the contribution due to purchases from individuals in class \( j \) from individuals in class \( i \) due to the production part of the individuals’ activity in class \( j \). Overall, the two effects together yield the total rate \( a_{ij} \) which represents the contribution of one unit of income in class \( j \) in period \( t \) to the income of class \( i \) in period \( t + 1 \).

From the matrix \( A \) we could derive all the macroscopic variables necessary for the analysis of the paper, such as the eigenvalue \( \lambda \), the growth rate \( r \), the entropy level \( H \) and so on. By looking at \( \Phi \) and \( \gamma \) we could determine whether the income process is fluctuating or steady and whether it is singular or diverse. By invoking the theorems to follow, this would allow us to deduce whether the economy has a tendency towards a higher or lower level of entropy and similarly for measures of inequality and social mobility. The next example allows agents from different classes to work in the same sector.

**Example 2.** Consider the following model, inspired from Alesina and Rodrik [5], of a one-sector economy with production function,

\[
Y(t) = K(t)^\alpha G(t)^{1-\alpha} L(t)^{1-\alpha}, \quad 0 < \alpha < 1,
\]

and multiple classes of agents (indexed by relative factor endowments) which we collapse into two, namely, those endowed with only capital (class 1) and those endowed with only labor (class 2), and where all agents have the same utility function,

\[
U = \int \log c(t) e^{-\rho t} dt, \quad \rho > 0.
\]

Moreover, there is a government sector which we subsume to the worker class, and which is funded through a capital tax: \( G(t) = \tau K(t) \); set \( L(t) = 1 \) for all \( t \). The income shares from the production activity can be written as \( \tilde{\alpha} = \alpha - \tau \alpha \) for class 1, and \( 1 - \tilde{\alpha} = 1 - \alpha + \tau \alpha \) for class 2, so that the actual income, consumption, and savings are given by \( y_1(t) = (\alpha \tau^{1-\alpha} - \tau) K(t) \), \( c_1(t) = \rho K(t) \), \( s_1(t) = y_1(t) - c_1(t) \), for class 1 and by \( y_2(t) = ((1 - \alpha) \rho^{1-\alpha} + \tau) K(t) \), \( c_2(t) = y_2(t) \), \( s_2(t) = 0 \) for class 2.\(^{14}\)

Notice that the agents in the economy jointly participate in the production of the (single) commodity and are therefore naturally linked through the processes of consumption and production. From the point of view of our framework, this means that we can write down an interaction matrix for this exchange economy that takes these flows into account. We therefore extend the original model of Alesina and Rodrik, which does not have any interaction matrix, and derive such a matrix \( A \) for the exchange economy described.

Let \( \tilde{c}_i = \frac{c_i(d)}{y_i(t)} \) be the proportion of income spent on consumption by individuals in class \( i \), for \( i = 1, 2 \). Income spent on consumption is divided to the two classes according to the shares of wages and capital income. Therefore, consumption expenditures by agents in class 1, ultimately constitute flows of income from class 1 to both class 1 and class 2; and similarly for class 2. We also record savings (only relevant for class 1) as flows from class 1 to itself. Multiplying these static flows by \( 1 + r \) to allow for growth yields the entries for the interaction matrix \( A \):

\[
a_{11} = (\tilde{c}_1 \tilde{a} + (1 - \tilde{c}_1))(1 + r), \quad a_{12} = \tilde{c}_2 \tilde{a}(1 + r), \quad a_{21} = \tilde{c}_1 (1 - \tilde{a})(1 + r), \quad a_{22} = \tilde{c}_2 (1 - \tilde{a})(1 + r),
\]

where \( \tilde{c}_1 = \frac{\rho}{\rho \tau^{1-\alpha} + \rho} \), \( \tilde{c}_2 = 1 \). Note also that, \( v = (v_1, v_2) = \left( \frac{\tilde{\alpha}}{\tilde{\alpha} \rho^{1-\alpha} + 1}, 1 \right) \).

\(^{14}\)More precisely, in the original model agents differ by their endowments in labor and capital; here we assume agents have either a fixed capital and zero labor endowment (class 1) or a fixed labor and zero capital endowment (class 2).
To give a more concrete idea, we can calculate the matrix $A$ and corresponding macroscopic parameters for the values $\alpha = 0.6$, $\rho = 0.4$, and $\tau = 0.02$, and obtain:

$$A = \begin{pmatrix} 0.87 & 0.54 \\ 0.20 & 0.53 \end{pmatrix}, \quad P = \begin{pmatrix} 0.81 & 0.19 \\ 0.51 & 0.49 \end{pmatrix},$$

with eigenvalue and (normalized) eigenvector (for $A$),

$$\lambda = 1.07, \quad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2.7 \\ 1 \end{pmatrix},$$

and macroscopic parameters,

$$r = 0.07, \quad \sigma^2 = 0.35, \quad H = 0.54, \quad \Phi = -0.48, \quad \gamma = 0.75.$$

From this, we deduce that the corresponding economy is constant and diverse, which in our framework means that, under a process of variation and natural selection, the evolutionary entropy will tend to increase and the income shares of the two classes will tend to get closer. In our model, this can happen in many ways that are picked up by “perturbations” of the matrix $A$. In particular, whether a more egalitarian or a more stratified society emerges depends on the evolutionary process through the mechanisms of variation and selection. This is a different yet complementary mechanism from the one of Alesina and Rodrik’s original model, where the role of the government and hence the amount of redistribution is decided through voting and depends on the distribution of the relative factor endowment ratios across individuals in the population. We view our approach as adding a further constraint or source of pressure towards more or less redistribution, as a function of the basic parameters $\Phi$ and $\gamma$ of the underlying steady state income generation process.

4 The Entropic Selection Theorem

In this section, we state and discuss the Entropic Selection Theorem (Theorem 1), which is central to our overall approach. It also constitutes a core result from directionality theory (Demetrius [20]). In our context, it relates the environment of the economy – its constancy and heterogeneity – to the evolution of the entropy measure $H$. In the next section we will further relate the entropy measure to economic measures of inequality and of social and income mobility. The following theorem therefore will have implications for redistribution within societies as well as its persistence. In a subsequent section, we also derive consequences concerning the robustness or resilience of higher entropy societies.

**Entropic Selection.** The Entropic Selection Principle is a local result. It pertains to the outcome of
competition locally between an original or ancestral population defined in terms of its mode of allocating resources, and a variant population defined in terms of a related mode. In any natural population and production process new variants are continually being introduced. At each instant the variant types must compete with the original types for the existing resources. A variant type will persist if it has a selective advantage or it will disappear if the original type has the selective advantage.

Consider a population with an income process evolving as in the framework of Section 2. The next theorem identifies a central variable, the evolutionary entropy $H$, as characterizing what types of interactions evolve and concomitantly what kind of social dispositions will proliferate. The question we ask here concerns the long-term change in evolutionary entropy as new variants arise.

Our analysis shows that whether higher or lower entropy interactions prevail, depends on characteristics of the underlying economic environment. Steady and diverse environments are conducive towards higher entropy interactions, while unsteady and singular ones are conducive towards lower entropy interactions. The following result states how the level of entropy of the underlying interaction will evolve globally in all possible cases.

**Theorem 1 (Entropic Selection Theorem).** The outcome of the selection process in a society evolving according to the income process $Y(t)$ described by Eq. (4) above is characterized by the following four cases:

(Ia) If the income process is constant and diverse ($\Phi < 0, \gamma > 0$), entropy tends to increase;

(Ib) If the income process is constant and singular ($\Phi < 0, \gamma < 0$), entropy tends to increase, provided total income is sufficiently large ($Y > \gamma/\Phi$); otherwise for small total income ($Y < \gamma/\Phi$) entropy increases with a probability that increases in the total level of income;

(IIa) If the income process is fluctuating and singular ($\Phi > 0, \gamma < 0$), entropy tends to decrease;

(IIb) If the income process is fluctuating and diverse ($\Phi > 0, \gamma > 0$), entropy tends to decrease, provided that total income is sufficiently large ($Y > \gamma/\Phi$); otherwise for small total income ($Y < \gamma/\Phi$) entropy decreases with a probability that increases in the total level of income.

This suggests that it is essentially in constant environments ($\Phi < 0$) that one should expect to find higher entropy societies. In large economies the result is general. When aggregate income is not sufficiently large ($Y < \gamma/\Phi$) then, to guarantee the same result, one needs that the environment also be diverse ($\gamma > 0$). Conversely, it is when the environment is fluctuating ($\Phi > 0$) that one should expect to find low entropy societies. Again, in large economies this is general. When aggregate income is not sufficiently large ($Y < \gamma/\Phi$) then, to guarantee the same result, one needs that the environment also be singular ($\gamma < 0$).

As we will see in the next section, when we relate the evolutionary entropy measure $H$ to a measure of income inequality (the Theil index $T$), this characterizes when to expect equal and when to expect unequal societies to prevail. It also suggests that when the economy is not sufficiently large, the outcome may be uncertain depending on the sign of the second-order variable ($\gamma$), which reflects the degree of heterogeneity of the income process. Notice that the model of cultural evolution implicit in our analysis is rather general and assumes neither an infinite population nor an infinite level of aggregate income.

The analytical basis for the Entropic Selection Theorem involves the integration of the ergodic theory of dynamical systems with the theory of diffusion processes (Demetrius [20]). The proof of the theorem is sketched below with some key steps proved in Appendix B.1. Before that, we offer the following intuition for the result.
Heuristics for the Entropic Selection Theorem. The basic intuition is that a society with high evolutionary entropy \( H \) is more stable and better suited for a constant environment. By intensifying its link structure, it can adapt itself better and extract more out of such an environment; at the same time it does not have the appropriate flexibility to adapt to a fluctuating environment. A society with low evolutionary entropy on the other hand is less stable and having a relatively more sparse link structure, it can adapt more easily to changes and thus extract relatively more from a fluctuating environment; at the same time it does not have the richness of pathways to do as well in a constant environment.

The mechanism operates at the individual level with individuals in different classes of a community or subpopulation interacting with individuals in the same and in other classes. Given an overall interaction structure \( A \), a successful variant interaction structure \( A^* \) that is adopted by a subpopulation can spread at the individual level to the whole society. The entropic selection result can be interpreted loosely as saying that in more constant environments, societies consisting of individuals with more cooperative and reciprocal traits will tend to prosper, whereas in fluctuating environments more selfish traits will tend to prosper.

We now sketch the argument more formally. Besides the first-order effect of the constancy of the environment, it also includes the effect of heterogeneity. Further details of the proof are in Appendix B.1.

Invasion Dynamics. The essence of our argument lies in the interaction between the aggregate production of a given population (the incumbent population, described by \( A \)) and that of the variant population (described by \( A^* \)). We model variants (mutants or invaders) as having their own production and redistribution technology represented by a matrix \( A^* \), which we model as being a perturbation \( A(\delta) = A^{1+\delta} \) of the original matrix \( A \), along with corresponding macroscopic parameters \( r^* = r(\delta), \sigma^2 = \sigma^2(\delta), H^* = H(\delta) \), for \( \delta \in \mathbb{R} \) small in absolute value. Their aggregate income follows a stochastic process described by,

\[
dY^*(t) = r^*Y^*(t)dt + \sigma^*\sqrt{Y^*(t)}dW^*(t),
\]

where \( W^*(t) \) is a Brownian motion independent of \( W(t) \). Therefore, if the total production is given by

\[
Z(t) = Y(t) + Y^*(t),
\]

then, setting \( \Delta r = r^* - r, \Delta \sigma^2 = \sigma^2 - \sigma^2 \), and defining the income share of the invaders as,

\[
p(t) = \frac{Y^*(t)}{Z(t)},
\]

one can look at the stochastic processes defined by

\[
dZ(t) = (r + p(t)\Delta r)Z(t)dt + \sigma\sqrt{1 - p(t)Z(t)}dW(t) + \sigma^*\frac{p(t)}{Z(t)}dW^*(t),
\]

and

\[
dp(t) = p(t)(1 - p(t))\left(\Delta r \frac{\Delta \sigma^2}{Z(t)} + \sigma p(t)\frac{1 - p(t)}{Z(t)}dW(t) + \sigma^*p(t)\frac{1 - p(t)}{Z(t)}dW^*(t),
\]

and solve for the process \( p(t) \). Assuming that total aggregate production is constant, \( Z(t) = Y \), then the
process \( p(t) \) can be shown to be a diffusion process with drift:

\[
\alpha(p(t)) = p(t)(1 - p(t)) \left( \Delta r - \frac{\Delta \sigma^2}{Y} \right)
\]

and variance:

\[
\beta(p(t)) = \frac{p(t)(1 - p(t))}{Y} \left( \sigma^2 p(t) + \sigma^2 (1 - p(t)) \right),
\]

and with a well-specified density. Notice that the process \( p(t) \) depends in an important way also on the variances of the respective processes \( Y(t) \) and \( Y^*(t) \).

Letting \( p_0 = p(0) \) denote the initial frequency of the mutant and \( \rho(p_0) \) the probability that the diffusion process leads to an absorption in the state \( p = 1 \) (extinction of the incumbent population), one shows that,

\[
\rho(p_0) = 1 - \left( \frac{1 - \frac{\Delta \sigma^2}{\sigma^2} p_0}{\frac{1 - \Delta \sigma^2}{\sigma^2} Y} \right)^{\frac{\Delta \sigma^2}{\sigma^2} + 1},
\]

where \( s = \Delta r - \frac{\Delta \sigma^2}{Y} \). The sign of the expression \( s \) is then crucial in determining whether a variant is successful in invading or not. The process \( p(t) \) is absorbed in 0 (invaders disappear) for any possible perturbation, if \( \Delta r < 0 \), \( \Delta \sigma^2 \geq 0 \), or \( \Delta r \leq 0 \), \( \Delta \sigma^2 > 0 \); that is, when \( s < 0 \) where we have \( \Delta r < \frac{\Delta \sigma^2}{Y} \) meaning that the incumbent’s growth rate advantage is greater than its normalized variance disadvantage; such an incumbent will prevail. On the other hand, the process \( p(t) \) is absorbed in 1 (invaders take over) for any possible perturbation, if \( \Delta r > 0 \), \( \Delta \sigma^2 \leq 0 \), or \( \Delta r \geq 0 \), \( \Delta \sigma^2 < 0 \); that is, when \( s > 0 \) where we have \( \Delta r > \frac{\Delta \sigma^2}{Y} \) meaning that the invader’s growth rate advantage is greater than its normalized variance disadvantage; in which case the invader will prevail.

From the definitions of the macroscopic parameters (see Appendix A.3), if \( \Phi \neq 0, \gamma \neq 0 \), and \( \sigma^2 \neq 0 \), we have the relations, \( \Delta r \approx \Phi \delta, \Delta \sigma^2 \approx \gamma \delta, \Delta H \approx -\sigma^2 \delta \), where \( \delta \in \mathbb{R}, \delta \approx 0 \), is the perturbation parameter. In order to express the selective advantage for the limiting case where \( \delta \to 0 \), we use the more general formula,

\[
s = -\left( \frac{\Phi}{\gamma} \right) \Delta H,
\]

where \( \Delta H = H^* - H \). This now describes the selective advantage directly in terms of the macroscopic parameters of the steady state income process \( Y(t) \) and of the entropy differential between the two populations \( (\Delta H) \). This covers a very wide class of models of cultural evolution.\(^\text{15}\)

\(^{15}\)An important aspect of the present approach is that it assumes neither an infinite (population or) level of aggregate production \( Y \to \infty \) nor an infinite amount of available resources \( X \to \infty \). Instead the latter two are special cases of the present, more general, approach. The following diagram summarizes the selective advantages corresponding to the different cases.

\[
s = -\left( \frac{\Phi}{\gamma} \right) \Delta H
\]

\[
Y \to \infty \quad X \to \infty
\]

\[
s = -\Phi \Delta H
\]

\[
X \to \infty
\]

\[
s = \Delta r - \frac{\Delta \sigma^2}{Y}
\]

\[
Y \to \infty
\]

The Malthusian case where the selective advantage \( s \) is based exclusively on the growth rate differential \( (\Delta r) \) between the two populations, is obtained in the limiting case where resources and (population or) total production are infinite (see Demetrius [20], Sections 2.3 and 6.3, for further discussion).
5 Evolutionary Entropy, Inequality, and Social Mobility

In this section, we establish the implications of the Entropic Selection Theorem for the evolution of inequality and social mobility of societies. In order to state our main results, Theorems 2 and 3, we first introduce measures of income inequality and social and income mobility which we relate formally to corresponding measures of evolutionary entropy.

Population Evolution and Social Mobility. So far we have introduced the social classes and the stochastic process representation of the aggregate population (10) and have not directly used the evolution of the population. To address issues of mobility and inequality, we discuss the process of population evolution in more detail. As with the processes of aggregate production \(Y(t)\) and resources \(X(t)\), we consider the population process, \(N(t) = \sum_{i=1}^{d} n_i(t)\), in steady state

\[
n(t + 1) = Bn(t),
\]

with associated population matrix,

\[
B = (b_{ij}), \quad b_{ij} \geq 0, 1 \leq i, j \leq d,
\]

where \(b_{ij}\) measures the rate at which individuals from class \(j\) contribute to individuals of class \(i\), which is to be interpreted as representing rates of how individuals transfer from one class to another. To simplify the analysis we maintain the assumption that the classes while having possibly different amounts of income, have the same population size, \(n_i(t) = N(t)/d\); this amounts to assuming that the rows have constant sum. Again, we are interested in the steady state distribution of the population across classes, where here the right eigenvector takes the form \(w = (w_1, \ldots, w_d) = \lambda(1, \ldots, 1) \in \mathbb{R}_+^d\), where \(\lambda \in \mathbb{R}_+\), is the dominant eigenvalue such that \(Bw = \lambda w\). The steady state share of population that is in class \(i\), \(i = 1, \ldots, d\), is always given by \(w_i = 1/d\). As above, \(\bar{r} = \log \lambda\) denotes the steady state growth rate of the population.

From this we can also define the Markov matrix which here takes the simpler form

\[
Q = (q_{ij}) = \left(\frac{b_{ij}w_j}{\lambda w_i}\right) = \lambda^{-1} B, \quad 0 \leq q_{ij} \leq 1, 1 \leq i, j \leq d.
\]

It describes the probabilities with which individuals move across classes. An element \(q_{ij} \geq 0\), can be interpreted as the probability that, in the steady state, an individual in class \(i\) came from class \(j\). Defining \(\rho = (\rho_1, \ldots, \rho_d)\) as the corresponding stationary distribution satisfying \(\rho Q = \rho\), we can define the entropy of the population

\[
\tilde{H} = -\sum_{i=1}^{d} \rho_i \sum_{j=1}^{d} q_{ij} \log q_{ij}, \quad 0 \leq \tilde{H} \leq \log d.
\]

To the extent that the \(d\) classes are social classes, or describe occupational choices, this measure of entropy is a natural measure of social mobility in that it measures precisely the extent and intensity of flows between the \(d\) classes. We can again derive the other macroscopic variables \(\tilde{r}, \tilde{\Phi}, \tilde{\sigma}^2, \tilde{\gamma}, \tilde{\kappa}\), which satisfy the analogous relations as the corresponding variables for \(A\). Moreover, the macroscopic variables associated to the population process are also positively correlated with the corresponding variables derived for the matrix \(A\). We come back to the evolution of social mobility in the next section.

Theil Index and Evolution of Income Inequality. We here establish a connection between certain
measures of entropy and the Theil index, a well-known entropy-based measure for the income distribution of an economy, that allows to decompose overall inequality into across-group and within-group inequality (see, e.g., Cowell [17]). In our case, we only consider across group inequality and we do not normalize the index to lie between $[0, 1]$. Theil’s index can then be written as,

$$T(v) = \frac{1}{d} \sum_{i=1}^{d} \frac{v_i}{\bar{v}} \log \left( \frac{v_i}{\bar{v}} \right), \quad 0 \leq T(v) \leq \log d,$$

where $\bar{v} = \sum_{i=1}^{d} v_i/d$ is the steady state average income of a class $i$; and $v$ is the income distribution corresponding to the eigenvector of $A$; we also write this as $T(0) = T(v)$, and write $T(\delta) = T(v(\delta))$ for the Theil index of the distribution of the eigenvector of the perturbed matrix $A(\delta)$. 

**Proposition 1 (Evolutionary Entropy and Inequality).** For perturbations of the form $A^{1+\delta}$, we have that the Theil index and the evolutionary entropy move in opposite directions, $\Delta T \Delta H < 0$, where $\Delta T = T(\delta) - T(0)$ and $\Delta H = H(\delta) - H(0), \delta \in \mathbb{R}$ small.

This result is crucial in linking inequality as measured by $T$ with the evolutionary entropy $H$.

Furthermore, we can define the following notion of income mobility (see Fields [26, 27]) that compares two (successive) income distributions, say $v = (v_1, \ldots, v_d)$ and $v' = (v'_1, \ldots, v'_d)$, defined by

$$E(v, v') = 1 - \frac{T(v + v')}{T(v)}.$$

As is discussed in Fields [26, 27], this is a measure of the degree of equalization of income distributions $v$ and $v'$, in the sense that positive values indicate that average income $\frac{v + v'}{2}$ are more equally redistributed than base income $v$ relative to the inequality measure $T$. Of course, income mobility can be defined with other inequality measures besides $T$, such as the Gini coefficient. We also write $E(\delta, 0) = E(v(\delta), v(0))$.

**Redistributive Selection.** We now turn to our main implication, which establishes a link between the nature of the economic environment and the evolution of inequality in the society. Given the last proposition, the notion of evolutionary entropy plays a central role in establishing the link. Again, in our evolutionary model, variants with different modes of redistribution (perturbed matrices $A^{*}$ with levels of inequality $T^{*}$) are continually introduced; these variants have to compete with the original types (operating with matrix $A$ with level of inequality $T$) for the existing resources. The outcome of the evolutionary process depends on which of the two types has a selective advantage. This in turn depends on the nature of the economic environment.

The following result contains our characterization of the evolution of inequality (measured by $T$).

**Theorem 2 (Redistributive Selection Theorem).** The outcome of the selection process facing a society evolving according to processes $Y(t)$ and $N(t)$ described by respectively Eqs. (4) and (13) above is characterized by the following four cases:

(Ia) If the income process is constant and diverse ($\Phi < 0, \gamma > 0$), income inequality tends to decrease;

(IIb) If the income process is constant and singular ($\Phi < 0, \gamma < 0$), income inequality tends to decrease, provided total production is sufficiently large ($Y > \gamma/\Phi$); otherwise for small total production ($Y < \gamma/\Phi$), income inequality tends to decrease with a probability that increases in total production;
If the income process is fluctuating and singular ($\Phi > 0, \gamma < 0$), income inequality tends to increase;

If the income process is fluctuating and diverse ($\Phi > 0, \gamma > 0$), income inequality tends to increase, provided total production is sufficiently large ($Y > \gamma / \Phi$); otherwise, for small total production ($Y < \gamma / \Phi$), income inequality will tend to increase, with a probability that increases in total production.

This result follows as a direct application of the Entropic Selection Theorem (Theorem 1) and Proposition 1. It shows that, whether equal or unequal societies tend to prevail depends on the constancy and diversity of the environment. More equal societies tend to emerge in constant and diverse environments, while less equal ones tend to emerge in fluctuating and singular ones. Moreover, more equal societies also grow faster in constant and diverse environments (than less equal ones), and, conversely, less equal societies grow faster in fluctuating and singular environments (than more equal ones). Thus, whether more equal or less equal societies grow faster can also depend on the constancy and heterogeneity of the underlying environment.

Income and Social Mobility. Next we relate our measures of income inequality ($T$), social mobility ($\tilde{H}$), and income mobility ($E$), introduced above to each other and to our central notion of evolutionary entropy ($H$); this readily yields a characterization of the selective advantage of societies, also in terms of income and social mobility for different environments.

Our measure of income mobility $E$ defined using the Theil index can be shown to be positively correlated with our measure of entropy $H$.

Proposition 2 (Evolutionary Entropy and Income Mobility). For perturbations of the form $A^{1+\delta}$, we have that income mobility as measured by $E$ and changes in the evolutionary entropy $H$ are positively correlated, $E \Delta H > 0$, where $\Delta H = H(\delta) - H(0)$, $\delta \in \mathbb{R}$ small.

In other words, small perturbations that increase the entropy $H$ will be associated with a positive income mobility $E$. Together with Proposition 1 this readily implies that changes in inequality are negatively correlated with income mobility, $E \Delta T < 0$, that is, along perturbations of the form $A^{1+\delta}$, changes in the Theil index and the income mobility measure are negatively correlated. This relationship has been recently called the “Great Gatsby Curve,” Corak [15], and we will come back to it in Section 7. Loosely speaking it suggests that more equal societies are associated with higher levels of income mobility; and vice versa, societies with higher levels of income mobility tend to be more equal. The key link between these two measures is given by the notion of evolutionary entropy of the income process which links to the entropy level of the population processes. Another way of stating this theorem is that a highly stratified society will tend to be less mobile and hence increasingly stratified.

Furthermore, given the positive (linear) relation $(F, f)$ assumed between the income and the population process, it immediately follows that the entropy measure $H$ and the social mobility measure $\tilde{H}$ are also positively correlated.

Proposition 3 (Evolutionary Entropy and Social Mobility). For perturbations of the form $A^{1+\delta}$, we have that social mobility as measured by the entropy measure $\tilde{H}$, and the evolutionary entropy $H$ move in the same direction, $\Delta \tilde{H} \Delta H > 0$, where $\Delta \tilde{H} = \tilde{H}(\delta) - \tilde{H}(0)$ and $\Delta H = H(\delta) - H(0)$, $\delta \in \mathbb{R}$ small.

In other words, small perturbations that increase the entropy $H$ will also increase social mobility as measured by $\tilde{H}$. Again, together with Proposition 1 this readily implies that changes in inequality are
negatively correlated with changes in social mobility, $\Delta T \Delta \tilde{H} < 0$. Thus, along perturbations of the form $A^{1+\delta}$, the Theil index of inequality and the social mobility measure move in opposite directions.

Together with the entropic selection theorem we can then show the following result on mobility.

**Theorem 3** (Income and Social Mobility Theorem). Societies with a constant and diverse income process ($\Phi < 0, \gamma > 0$) will tend towards higher levels of income and social mobility.

In constant and diverse environments more mobile societies will tend to prevail, while in fluctuating and singular ones less mobile and more stratified societies will tend to prevail. The main result of the next section (Theorem 4) shows that more stratified societies are more vulnerable to shocks and thus more fragile than more egalitarian ones.

### 6 Economic and Social Robustness and Inequality

Do equal societies present any advantage over unequal ones? A recent book by Wilkinson and Pickett [50], drawing from evidence carried out by a large number of researchers, suggests that economic inequality is “socially corrosive” in that it is positively correlated with a variety of “undesirable” indices that cover issues ranging from health and mental disorders, life expectancy and literacy rates to fairness, trust, and happiness of individuals in society. Stiglitz [46, 47] provides further economic arguments in favor of lower levels of inequality, with special attention to the case of the US.

A property which seems to distinguish stratified from egalitarian societies is the degree of resilience or robustness of the society. We here use the term resilience to describe the capacity of the society to return to its steady state condition when subject to a variety of instantaneous shocks which may be demographic or economic. *Demographic shocks* include: the spread of an epidemic, a situation which may impose large strains on healthcare facilities; the effect of an earthquake or other natural disaster which involves a large loss of lives and disruption of the economy; military conflicts which inflict a large toll on young individuals in the populations. *Economic shocks* include: a macroeconomic recession, a situation which may lead to a reorganization of the workforce; financial collapse of large banks, a condition which can impact a large spectrum of economic projects; a collapse in the price of a commodity, such as oil which is critical for the economy of the country.

Observation in Europe and other countries across the world reveals a large diversity in the response to such shocks. The capacity of the socio-economic community to remain functional in spite of such shocks appears to be highly correlated with the degree of economic inequality.

We formalize the notion of robustness or resilience and appeal to what we call a Perturbation Stability Theorem to furnish an explanation of the observed correlation between economic inequality and the resilience of the community.

**Robustness.** At a general level, robustness is associated to the invariance of a society’s macroscopic variables in the face of endogenous or exogenous perturbations to the system. The way this is captured by directionality theory is by means of the formalism of large deviation theory (see Demetrius [20] and Demetrius et al. [21]).

Formally speaking, our robustness theorem focuses on the macroscopic parameter $\Phi$, and introduces the probability $P_\epsilon(n)$ that sample averages (say $\tilde{\Phi}(n)$) of sample trajectories of length $n$, differ by more

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16 As is clear from the Appendix, A.3, this variable is the average of the potential function $\varphi$, which is central in determining all other macroscopic parameters of the process $Y(t)$, including the growth rate and the variance. It is for this reason that the robustness measure is based on the probability of sample averages approaching the true value of $\Phi$. 
than $\epsilon$ from ensemble averages over all trajectories, for fixed $\epsilon > 0$. The ergodic theorem states that $P_\epsilon(n)$ converges to zero for large enough sample lengths; moreover, the convergence rate is known to be at least exponentially fast.\(^{17}\) Hence, as a measure of robustness $R$, we can use the following fluctuation decay rate,

$$R = \lim_{n \to \infty} \left[ -\frac{1}{n} \log P_\epsilon(n) \right].$$

(18)

Large values of $R$ correspond to fast rates of convergence of the macroscopic variable $\Phi$ to the steady state values; small values of $R$ correspond to slow rates of convergence. Thus, $R$ characterizes the adjustment rate of the (fundamental) macroscopic observable $\Phi$ in the face of shocks in the underlying system. More generally, because all further macroscopic variables are directly determined through the variable $\Phi$, the measure $R$ provides a measure of the convergence rate of all the macroscopic variables to their steady state values. The complexity stability theorem asserts that changes in robustness are positively correlated with changes in evolutionary entropy, $\Delta H \Delta R > 0$.

**Theorem 4** (Perturbation Stability Theorem). Societies with higher evolutionary entropy tend to be more resilient to disruptions and perturbations and also tend to return faster to the steady state equilibrium.

In view of Proposition 1 this can be restated as saying that changes in robustness are negatively correlated with changes in inequality, $\Delta T \Delta R < 0$. This readily implies the following statement.

**Proposition 4** (Redistribution and Perturbation Stability). Societies with less income inequality tend to be more resilient to disruptions and perturbations and also tend to return faster to the steady state equilibrium.

These results point to one significant advantage of societies with more equal redistribution, namely a typically higher level of robustness to shocks. This has important implications for the expected lifespan or survival rate of societies, in the following sense. The collapse of a society may be associated to a particularly large deviation from the true value $\Phi$ and this is more likely to occur for a society with less equal redistribution.

### 7 Empirical Evidence

Assuming certain basic properties and relations between the different aggregate processes $N(t)$ and $Y(t)$ our theory derived several testable implications. While we plan to test these hypotheses empirically in future work, we here present some evidence for the three main results of the paper, taken from existing literature.

**Macroeconomic Volatility and Income Inequality.** The central relation derived in the paper relating inequality to fluctuations of the resource process (Theorem 2) finds some empirical validation in the literature studying macroeconomic fluctuations and income distribution. Breen and García-Peñalosa [12] study the impact of macroeconomic volatility (measured as the standard deviation of the growth rate of real GDP per capita) on income distribution (measured by the Gini coefficient) for a large set of countries:\(^{18}\) Laursen

\(^{17}\)As mentioned in Appendix B.4, it can be shown that there exist constants, $c_0, c_1 > 0$, such that, $P_\epsilon(n) \leq c_0 \exp^{-c_1 n}$.

\(^{18}\)Our theoretical measure for variability/constancy of the environment is derived from the variable $\Phi$, which measures the difference between the steady state growth rate and the rate at which the macroscopic variables return to steady state values; the larger $\Phi$ the more erratic the process and the larger its volatility measured in terms of the standard deviation of the year to year growth rate. An economy with a very low (negative) $\Phi$ will exhibit a year to year growth rate typically very close to

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and Mahajan [34] study similar effects for different subsets of countries using the income share of the bottom quintile of the population as inequality measure; more recently, Huang et al. [31] study macroeconomic fluctuations and inequality in the US. All these papers find a positive and significant correlation between macroeconomic volatility and income inequality, which is consistent with our theory, linking fluctuations of the environment to fluctuations of aggregate production, through the evolutionary entropy, and this in turn to income inequality. Neither paper looks at the role of the diversity or singularity of the environment and neither paper looks at the resource process strictly speaking. A systematic study of these relationships, also taking into account the diversity of the environment, is deferred to future work. See Figure 3, taken from Breen and Garcia-Peñalosa [12].

In the paper, we have formally that our notion of evolutionary entropy which is at the center of the formal derivations is positively correlated with income inequality as well as social and income mobility. We have also argued somewhat informally that it is also a measure of the level of cooperativeness of society. Thus we expect societies with high volatility to exhibit low levels of cooperativeness, high levels of selfishness, and societies with low volatilities to have high levels of cooperativeness, low levels of selfishness. Somewhat related to cooperativeness is the measure of trust. Consistent with our overall approach is the finding of a negative relationship between volatility and trust in Sangnier [43].

argued informally that our notion of evolutionary entropy which is at the center of the formal derivations

**Social Mobility and the Great Gatsby Curve.** The *Great Gatsby Curve*, plotting intergenerational income elasticity on income inequality, was highlighted recently in a speech delivered by the chairman of the US Council of Economic Advisors, Alan Krueger, who using work by Corak [15], pointed to a positive relation between higher income inequality and higher intergenerational income elasticity in the United States; while one with a large (positive) Φ will exhibit a less stable year to year growth rate. In the absence of an empirical measure for Φ we think the volatility measure used in this and other papers are reasonable proxies. In Appendix A.3 we also formally show a positive correlation of Φ with the measure of volatility $V_0$. 

![Figure 3: Income inequality and macroeconomic volatility, averaged by region, [12].](image)
States and in twelve developed economies (see also the 2012 US Economic Report of the President [16]). This is very much consistent with our theory that higher income inequality reflects a lower evolutionary entropy of the resource process and hence a lower entropy of the population process which is directly a natural measure of social mobility across occupational classes. To the extent that incomes in different classes are relatively stable, our measure should be strongly positively correlated to intergenerational income elasticity and other measures of income mobility. See Figure 4, taken from 2012 US Economic Report of the President [16].

Inequality and Resilience to Shocks. The role of inequality in affecting the resilience of a country to respond to shocks was an important topic of a 2011 report of the United Nations Development Programme. In the report the authors point out that indeed less equal societies are slower in adapting to shocks; they further show that these are also the countries with less stable environments. Kahn [32] shows that national income but also the level of income inequality has an important effect on the death toll of natural disasters.

8 Conclusions

Discussions regarding the phenomenon of economic inequality, its origin and spread, have moved recently from the confines of specialized academic departments to being among the most debated topics of the public at large. At least three issues appear to fuel the debate, namely, the empirical reality that the gap between rich and poor has shown a remarkable increase in a number of countries in the last 30 years (Piketty [40]); the empirical observation that countries with large economic inequality have highly fragile infrastructure, a high degree of corruption and dysfunction in a number of canonical indices of health and good citizenship (Wilkinson and Pickett [50]); the political problems which emerge in efforts to arrest the flow of immigrants from highly stratified societies with unstable economies.

The theory developed in this paper contributes to understanding the origin and spread of inequality
by recognizing the importance of the macroeconomic environment in driving income distribution. The intermediary element connecting the macroeconomic environment and the income distribution are the social preferences of the individuals in the communities (reflected in what we call the interaction matrix, or the society’s intertemporal production function). The theory shows that steady and diverse environments induce more equal redistribution, mediated through a higher degree of cooperativeness (higher entropy). Fluctuating and singular environments favor less equal distribution mediated through a higher degree of self-serving behavior (lower entropy). Furthermore, the environment through its effect on the evolutionary entropy will also impact both intergenerational social mobility and the resilience of the economies in such a way that reinforces and amplifies the original effect, thus further explaining the coexistence of egalitarian societies on one side and highly stratified ones on the other.

Regulation, fiscal, and monetary policy may be designed to take this into account. Indeed the theory provides a new rationale for macroeconomic stabilization policies. The case for such policies has been rather discredited by new classical models of macroeconomics, with Lucas famously putting the cost of fluctuations at “something less than one tenth of a percentage point” of average consumption (Lucas [35], p. 27; also see Krusell et al. [33] for a more recent revised estimate). On the other hand, consistent with our viewpoint, the German law of 1967, “Gesetz zur Förderung der Stabilität und des Wachstums der Wirtschaft,” (StabG), explicitly identifies the macro-economic stability as a prime objective of economic policy and public finance;¹⁹ somewhat related is the United States “Full Employment and Balanced Growth Act of 1978,” strengthening the preceding “Full Employment Act of 1946.” Further empirical work is called for to test in more detail the different mechanisms and correlations derived in this paper. Hopefully this will provide a fuller understanding of the role of macroeconomic volatility and diversity in long-term processes and to better evaluate the role of redistribution and stabilization policies for society.

¹⁹Former Chancellor Helmut Schmidt actually elevated the objective of a stable economy to a cornerstone of West Germany’s national and international strategic policy, with the further objective of exporting such a policy to neighboring European countries (see, e.g., his speech of November 30, 1978, to the Bundesbank’s board of directors). With a Gini coefficient of around 0.25, West Germany exhibited relatively low inequality throughout the 1980’s before its unification in 1990.
References


APPENDIX

A Steady State

We here build on [6] to sketch some basic properties of the macroscopic variables of our dynamic system at steady state, as well as some properties of the perturbations of that system. This allows us to better understand the terminology and connections between the different variables in the model. These facts are derived in detail in the mentioned article using the formalism of random dynamical systems and statistical mechanics. While we can only limit ourselves to sketching the main steps, we refer to that article for a complete discussion; see also [20] and [22].

A.1 Random Dynamical Systems

We assume that the (possibly nonlinear) dynamic system

\[ v(t + 1) = A(t)v(t) \]

evolves to a steady state. At steady state we assume the process is represented by a constant \( d \times d \) matrix \( A = (a_{ij}) \) with \( a_{ij} > 0 \). As explained above, the entries \( a_{ij} \) are rates that represent averages. In fact, transfers from one class to the other are assumed to occur in discrete units and with fixed probabilities. Let \( D = \{1, 2, \ldots, d\} \) and define the set of all possible infinite backward sequences

\[ X = \prod_{\nu = -\infty}^{\infty} D_{\nu}, \quad \text{where} \ D_{\nu} = D. \]

We can then define the space of all possible infinite backward sequences

\[ \Omega = \{x \in X : a_{x_{\nu+1}x_{\nu}} > 0\}, \]

which we also refer to as the interaction space. It represents the space of all infinite backward paths of the graph associated with the matrix \( A \). These can be thought of following through the past transitions of a given “representative” unit of the numeraire commodity (e.g., a “euro coin”) between different individuals of different classes.

Let \( \tau : \Omega \to \Omega, (x_k) \mapsto (\tilde{x}_k) \), where \( \tilde{x}_k = x_{k+1} \) be the shift map, and let \( \mathcal{M} \) denote the set of probability measures that are invariant under the shift map \( \tau \). Defining \( \mu \) as the natural Markov measure on \( \Omega \) at the steady state, one can show that (see [22], Theorem 4.2) this is the unique probability measure that maximizes

\[ H_{\mu}(\tau) + \int \varphi d\mu \]

such that,

\[ \log \lambda = \sup_{\mu \in \mathcal{M}} \left\{ H_{\mu}(\tau) + \int \varphi d\mu \right\}, \tag{19} \]

where \( H_{\mu}(\tau) \) is the Kolmogorov-Sinai entropy for the system \( (\Omega, \mu, \varphi) \) and the function \( \varphi : \Omega \to \mathbb{R} \) is given by

\[ \varphi(x) = \log a_{x_1x_0}. \]

Analytically, it can be described explicitly by means of the Markov matrix \( P = (p_{ij}) \), where \( p_{ij} = \frac{a_{ij}}{\lambda v_i} \).
and \( v = (v_i) \) is the right eigenvector corresponding to the largest eigenvalue \( \lambda \) of the matrix \( A \); (see [6] and also [22], Theorem 4.2).

Furthermore, one can show using a generalization of the Central Limit Theorem (see [22], Theorem 7.1) that, under the Markov measure \( \mu \), for the dynamical system defined by Eq. (4), asymptotically, the deviations from the mean of appropriately constructed sample paths as \( n \to \infty \) can be approximated by a Brownian motion with variance \( \sigma^2 t \). Taking time limits, this yields a continuous time process with growth rate \( r \) and variance \( \sigma^2 Y(t) \) that can be represented as the diffusion given by Eq. (7); (see [22], Section 7).

### A.2 Perturbations

Throughout the paper we make use of perturbations of the interaction matrix of the form \( A(\delta) = A^{1+\delta} \). We here sketch a motivation.

Consider two dynamic systems at steady state given respectively by \((\Omega, \mu, \varphi)\) and \((\Omega, \mu^*, \varphi^*)\), where to capture the fact that say the latter is a mutation of the former, we assume that

\[
\varphi^* = \varphi(\delta) = \varphi + \delta \psi,
\]

with

\[
\int \varphi d\mu = \int \psi d\mu \quad \text{and} \quad \frac{d}{d\delta} \int \varphi d\mu |_{\delta=0} = \frac{d}{d\delta} \int \psi d\mu |_{\delta=0}.
\]

The first condition says that the deviation \( \psi \) of the mutant population has the same reproductive potential as the incumbent population; the second condition says that it changes in the same direction. This is sufficient for our results. However, for ease of representation and to simplify the analysis we consider the special case \( \psi = \varphi \), which, if we assume \( \varphi = \log a_{ij} \) (for corresponding realization \( a_{x_1x_0} = a_{ij} \)), implies

\[
\varphi(\delta) = \varphi + \delta \varphi = (1 + \delta) \log a_{ij} = \log a_{ij}^{1+\delta},
\]

which corresponds to perturbations of the interaction matrix of form

\[
A(\delta) = (a_{ij}^\delta) \equiv A^{1+\delta},
\]

considered throughout the paper. A perturbation where \( \delta > 0 \) (\( \delta < 0 \)) corresponds to a mutant population with lower (higher) entropy \( H_{\mu^*} \) as compared to the entropy of the resident population \( H_{\mu} \), where \( \delta = 0 \). See [22] for further discussion.

### A.3 Macroscopic Variables

Applying statistical mechanics tools to the dynamical system described in the previous section shows how to derive macroscopic variables from the function \( \varphi \) and the measure \( \mu \). These provide more concise description of the dynamical system involved.

**Growth Rate.** Consider the function

\[
S_n \varphi(x) = \sum_{k=0}^{n-1} \varphi(x^kx) = \sum_{k=0}^{n-1} \log a_{x_k+1x_k}
\]
so that denoting for given \((x_0, x_1, \ldots, x_n)\) by \(x^*\) any point in \(\Omega\) with \(x^*_i = x_i\), for \(i = 0, 1, \ldots, n\), then we have
\[
Z_n(\varphi) = \sum_{(x_0, x_1, \ldots, x_n)} \exp S_n \varphi(x^*) = \sum_{(x_0, x_1, \ldots, x_n)} a_{x_1 x_0} a_{x_2 x_1} \cdots a_{x_n x_{n-1}},
\]
so that, using the Perron-Frobenius Theorem, in the limit we have
\[
\lim_{n \to \infty} \frac{1}{n} \log Z_n(\varphi) = \log \lambda = r,
\]
which exists under general conditions on \(\varphi\). From the variational principle (19) mentioned above, we have
\[
r = \Phi + H,
\]
where \(H = H_\mu(\tau)\) denotes the Kolmogorov-Sinai entropy of \((\Omega, \mu, \varphi)\) and \(\Phi = \int \varphi d\mu\) denotes the reproductive potential or the mean energy associated with the (potential) function \(\varphi\); (see [6], [20]).

**Reproductive Potential and other Moments.** From the growth rate we can generate further macroscopic variables. Let \(\lambda(\delta)\) denote the dominant eigenvalue of the perturbed matrix \(A(\delta) = A^{1+\delta} = (a_{ij}^\delta)\), and \(r(\delta) = \log \lambda(\delta)\), for \(\delta \in \mathbb{R}\). Then as shown in [22], we have,
\[
r(\delta) = r(0) + \delta r'(0) + \frac{\delta^2}{2!} r''(0) + \frac{\delta^3}{3!} r'''(0) + \ldots,
\]
where
\[
\begin{align*}
r'(0) &= \lim_{n \to \infty} \frac{1}{n} \mathbb{E}_n (S_n \varphi) = \int \varphi d\mu = \Phi, \\
r''(0) &= \lim_{n \to \infty} \frac{1}{n} \mathbb{V}_n (S_n \varphi) = \sigma^2, \\
r'''(0) &= \lim_{n \to \infty} \frac{1}{n} \mathbb{E}_n \left[ S_n \varphi - \mathbb{E}_n S_n \varphi \right]^3 = \kappa.
\end{align*}
\]
Here \(\mathbb{E}_n\) and \(\mathbb{V}_n\) denote the expectation and variance with respect to the measure \(\mu_n\) on finite sequences of length \(n\) of the form \((x_0, x_1, \ldots, x_n)\), which is defined by
\[
\mu_n = \frac{S_n \varphi(x)}{\sum_{(x_0, x_1, \ldots, x_n)} S_n \varphi(x)}.
\]
Besides giving the reproductive potential \(\Phi\), this further gives the demographic variance \(\sigma^2\), and the correlation index \(\kappa\); clearly we can also write \(\Phi = \frac{dr(\delta)}{d\delta} \big|_{\delta=0}\).

Consider now the variance \(\sigma^2(\delta)\) obtained by further perturbing \(A(\delta)\) again. Then we can define
\[
\gamma = \frac{d\sigma^2(\delta)}{d\delta} \big|_{\delta=0},
\]
which is referred to as the demographic index; (as it can be used to approximate the demographic variance).

**Relations of the Macroscopic Variables.** We here summarize some of the relationships that hold between the macroscopic variables defined so far; (see [22] for derivations):

(a) \(\Phi = r - H\)
In our evolutionary analysis, an incumbent population is in competition with an invader population, which we capture in terms of a dynamic interaction between the two populations. The incumbent and the invader population steady state dynamics are given respectively by \((\Omega, \mu, \varphi)\) and \((\Omega, \mu^*, \varphi^*)\), where to capture the fact that the latter is a mutation of the former, as discussed in A.2, we assume that

\[ \varphi^* = \varphi(\delta) = \varphi + \delta \varphi, \]

which corresponds to an interaction matrix of the invader population of the form \(A(\delta) = A^{1+\delta}\).

We can then determine the macroscopic variables for the invader population as with the incumbent population, so that setting

\[ r^* = r(\delta), \sigma^{*2} = \sigma^2(\delta), \text{ and } H^* = H(\delta), \]

we get:

\[ \Delta r = r(\delta) - r(0) \approx \Phi \delta \]

\[ \Delta \sigma^{2} = \sigma^2(\delta) - \sigma^2(0) \approx \gamma \delta \]

\[ \Delta H = H(\delta) - H(0) \approx -\sigma^2 \delta. \]

For small \(\delta \in \mathbb{R}\), this readily gives the following relations:

\[ (f) \Phi < 0 \iff \Delta r \Delta H > 0 \]

\[ (g) \gamma > 0 \iff \Delta \sigma^{2} \Delta H < 0. \]

These will play an important role in the derivation of the main results.

**Interaction between Environment, Population, and Production.**

Throughout the paper we refer to the variable \(\Phi\) as representing the steadiness of the environment or of the income process. This was motivated in Section 2 by referring to the relationship \(\Phi = r - H\). In Section 7, discussing the empirical evidence, we also associated \(\Phi\) loosely to what is called macroeconomic volatility. We here make the connection more formal. Recall that \(\lim_{n \to \infty} \frac{1}{n} \log Z_n(\varphi) = r\) and consider the following notion of *volatility*,

\[ V_n = \mathbb{E}_n\left( \frac{1}{n} \log Z_n(\varphi) - r \right)^2, \]

defined for any \(n\) sufficiently large. Then it can be shown that \(V_n\) is positively correlated with the reproductive potential, that is, for \(n\) sufficiently large,

\[ (h) \Delta V_n \Delta \Phi > 0, \]

for \(\delta \in \mathbb{R}\) small and for \(\Phi \in \mathbb{R} \setminus [-\sigma^2, 0]\). To see this, notice that

\[ \frac{\partial}{\partial \delta} \left[ \log Z_n(\varphi(\delta)) \right] = \frac{\sum_{(x_0, x_1, \ldots, x_n)} (a_{x_1x_0} a_{x_2x_1} \cdots a_{x_nx_n-1})^{1+\delta} \log (a_{x_1x_0} a_{x_2x_1} \cdots a_{x_nx_n-1})}{\sum_{(x_0, x_1, \ldots, x_n)} (a_{x_1x_0} a_{x_2x_1} \cdots a_{x_nx_n-1})^{1+\delta}} \]

and that for \(n\) sufficiently large, we have \(\Phi > 0 \iff \log(a_{x_1x_0} a_{x_2x_1} \cdots a_{x_nx_n-1}) > 0\).

Environmental variables are in correspondence with both population and production variables, so that changes in first-order and second-order moments are strongly positively correlated at the steady state equilibrium. This plays a central role in linking our resource process to both the population and the production process. We capture this by assuming,
(i) $\Delta r \Delta \tilde{r} > 0$,

(j) $\Delta \sigma^2 \Delta \tilde{\sigma}^2 > 0$,

which in particular implies,

(k) $\Delta H \Delta \tilde{H} > 0$.

These relations play a key role in establishing the link between social mobility and income inequality in this framework.

B Proofs

B.1 Proof of Theorem 1

We here provide a sketch of the main steps. For a more detailed proof, we refer the reader to [22].

Let $Z(t) = Y(t) + Y^*(t)$ denote total aggregate production. The share of aggregate production of the invader population can be written as, $p(t) = \frac{Y^*(t)}{Z(t)}$. We are concerned with the evolution of this ratio.\(^{20}\)

The densities of $f(Y, t)$ and $f^*(Y^*, t)$ of the aggregate production levels $Y(t)$, and $Y^*(t)$ are characterized respectively by the Fokker-Planck equations

$$\frac{\partial f}{\partial t} = -r \frac{\partial (fY)}{\partial Y} + \frac{\sigma}{2} \frac{\partial^2 (fY)}{\partial Y^2} \quad \text{and} \quad \frac{\partial f^*}{\partial t} = -r^* \frac{\partial (f^* Y^*)}{\partial Y^*} + \frac{\sigma^*}{2} \frac{\partial^2 (f^* Y^*)}{\partial Y^*^2}.$$ 

Equivalently, we can characterize $Y(t)$ and $Y^*(t)$ respectively as the solutions to the stochastic differential equations

$$dY(t) = rY(t)dt + \sigma \sqrt{Y(t)}dW(t), \quad \text{(20)}$$

and

$$dY^*(t) = r^* Y^*(t)dt + \sigma^* \sqrt{Y^*(t)}dW^*(t), \quad \text{(21)}$$

where the processes $Y(t)$ and $Y^*(t)$ evolve simultaneously and stochastically independently, so that $W(t)$ and $W^*(t)$ are independent Brownian motions.

Let $A^* = A(\delta) = A^{1+\delta}$ represent the interaction matrix of the invader population with corresponding macroscopic variables $r^* = r(\delta), \sigma^* = \sigma^*(\delta), H^* = H(\delta), \Phi^* = \Phi(\delta), \gamma^* = \gamma(\delta), \kappa^* = \kappa(\delta)$.

It can be shown (see [22], Theorem 7.2) that equations (20) and (21) are equivalent to the system of stochastic differential equations,

$$dZ(t) = (r + p(t) \Delta r) Z(t)dt + \sigma \sqrt{1 - p(t)Z(t)}dW(t) + \sigma^* \sqrt{p(t)Z(t)}dW^*(t), \quad \text{(22)}$$

and

$$dp(t) = p(t)(1 - p(t)) \left( \Delta r - \frac{\Delta \sigma^2}{Z(t)} \right) dt - \sigma p(t) \sqrt{\frac{1 - p(t)}{2Z(t)}}dW(t) + \sigma^* (1 - p(t)) \sqrt{\frac{p(t)}{Z(t)}}dW^*(t). \quad \text{(23)}$$

\(^{20}\) Initially, the share $p(t)$ is small and the two populations evolve independently of each other. The invader population can be seen as drawing from resources not used or available to the incumbent. Then, as the invader population grows, the two populations compete for resources. We assume the two populations are in steady state assuming indirectly that the convergence to steady state is much faster than the selection process. This also justifies focusing on the case where the overall production is fixed ($Z(t) = Y$); see also [22], Section 2.
We need to solve this for the process \( p(t) \). Assuming total aggregate production is constant, \( Z(t) = Y, \)
then the process \( p(t) \) can be shown to be a diffusion process with drift
\[
\alpha(p(t)) = p(t) (1 - p(t)) \left( \Delta r - \frac{\Delta \sigma^2}{Y} \right)
\]
and variance
\[
\beta(p(t)) = \frac{p(t)(1 - p(t))}{Y} \left( \sigma^2 p(t) + \sigma^*^2 (1 - p(t)) \right);
\]
and that the process \( p(t) \) has density \( \psi \) solving the Fokker-Planck equation (see [22], Theorem 7.3),
\[
\frac{\partial \psi}{\partial t} = -\frac{\partial \left[ \alpha(p) \psi \right]}{\partial p} + \frac{1}{2} \frac{\partial^2 \left[ \beta(p) \psi \right]}{\partial p^2},
\]
with natural boundary conditions, \( \psi(0, t) = 0, \psi(1, t) = 1 \), that correspond to the cases \( p = 0 \) (when the invader population becomes extinct) and \( p = 1 \) (when the incumbent population becomes extinct). Notice that we set \( \alpha(p) \equiv \alpha(p, Y) \) and \( \beta(p) \equiv \beta(p, Y) \), so that \( \alpha(0) = \alpha(1) = 0 \) and \( \beta(0) = \beta(1) = 0 \). This implies a unique solution for any initial value \( \psi(p, 0) \).

Letting \( p_0 = p(0) \) denote the initial frequency of the mutant and \( \rho(p_0) \) the probability that the diffusion process leads to an absorption in the state \( p = 1 \) (extinction of the incumbent population), appealing to the backward Kolmogorov equation,
\[
\frac{\partial \psi}{\partial t} = \alpha(p) \frac{\partial \psi}{\partial p} + \frac{1}{2} \beta(p) \frac{\partial^2 \psi}{\partial p^2}
\]
and integrating, one shows that the invasion probability \( \rho(p_0) \) can be written as,
\[
\rho(p_0) = \frac{1 - \left( 1 - \frac{\Delta \sigma^2}{\sigma^2} p_0 \right)^{\frac{2Y S}{\Delta \sigma^2} + 1}}{1 - \left( 1 - \frac{\Delta \sigma^2}{\sigma^2} \right)^{\frac{2Y S}{\Delta \sigma^2} + 1}},
\]
where \( S = \Delta r - \frac{\Delta \sigma^2}{\sigma^2} \) (again, see [22], Section 7). The sign of the expression \( S \) thus becomes crucial in determining whether a variant is successful in invading or not. Except for the degenerate case of \( \frac{2Y S}{\Delta \sigma^2} + 1 = 0 \), \( \rho'(\cdot) \neq 0 \), and it is easy to show that convexity or concavity of \( \rho(\cdot) \) is determined by \( S \) alone, namely,
\[
S > 0 \Rightarrow \rho(\cdot) \text{ is convex}, \quad S < 0 \Rightarrow \rho(\cdot) \text{ is concave}.
\]
The exact curvature of \( \rho(\cdot) \) then depends on the magnitude of \( S \) and hence on the values of \( \Delta r, \Delta \sigma^2, \) and \( Y \). The exact relations between these variables in determining the sign of \( S \) and their effect on the invasion probability provides the conditions under which an invader’s level of entropy should be higher or lower than \( H \) in order to be successful.

Now, consider initial values \( p_0 \) close to zero, then the solution \( p(t) \) is absorbed in state \( p = 0 \) (extinction of the invader population) for any small perturbation, if
\[
\Delta r < 0, \Delta \sigma^2 \geq 0 \quad \text{or} \quad \Delta r \leq 0, \Delta \sigma^2 > 0. \quad (24)
\]

\(^{21}\)Strictly speaking, we need only to assume that this holds for \( t > t_0 \) for some \( t_0 \) that represents the instant where the exploitation competitive interaction between incumbent and invader population begins; this is consistent with the case we consider, where resources are finite and limited. See [22], Section 2, for further discussion on this point.
Under these conditions, one of the following two cases occurs,

(I) $\Phi < 0, \gamma \geq 0$, or $\Phi \leq 0, \gamma > 0$;

(II) $\Phi > 0, \gamma \leq 0$, or $\Phi \geq 0, \gamma < 0$.

In case (I), condition (24) for all perturbations is equivalent to $\Delta H < 0$ (the invader population has lower entropy; and the incumbent population with higher entropy takes over); in case (II), it is equivalent to $\Delta H > 0$ (the invader population has higher entropy; and the incumbent population with lower entropy takes over); (see [22], Theorem 7.4). This yields the more general formula for the selective advantage

$$S = -\left(\Phi - \frac{\gamma}{Y}\right) \Delta H,$$

where $\Delta H = H^* - H$.

In the limit, as $Y \to \infty$, the diffusion equation for $p$ degenerates to a linear differential equation, and the convexity criterion in terms of $S$ reduces to the growth rate differential $\Delta r$. In this case, we have, $\Phi < 0 \iff \Delta H < 0$ and $\Phi > 0 \iff \Delta H > 0$, and the reproductive potential alone determines the selective advantage of the level of entropy (again, see [22], Section 7).

**B.2 Proof of Proposition 1**

Consider the following further measure of entropy,

$$H_Y = -\sum_{i=1}^{d} s_i \log s_i, \quad 0 \leq H_Y \leq \log d,$$

where $s_i = v_i / \sum_{j=1}^{d} v_j$ is the steady state share of income of class $i$. Then it is easy to see that $H_Y$ and $T(v)$ are related as follows,

$$T(v) = \log d - H_Y,$$

so that $\Delta H_Y \Delta T < 0$.

We need to show that $\Delta H_Y \Delta H > 0$. To see this, consider the following matrix

$$A_Y = \begin{pmatrix}
    v_1 & \cdots & v_1 \\
    \vdots & \ddots & \vdots \\
    v_d & \cdots & v_d
\end{pmatrix},$$

where $v$ is the right eigenvector of $A$. Then it is easy to see that it is also a right eigenvector of $A_Y$, $A_Y v = \lambda_Y v$,

where $\lambda_Y = \sum_{i=1}^{d} v_i$. Moreover, the corresponding Markov matrix takes the form

$$P_Y = \begin{pmatrix}
    \frac{v_1}{\lambda_Y} & \cdots & \frac{v_d}{\lambda_Y} \\
    \vdots & \ddots & \vdots \\
    \frac{v_1}{\lambda_Y} & \cdots & \frac{v_d}{\lambda_Y}
\end{pmatrix},$$
such that the corresponding entropy satisfies
\[ H_Y = - \sum_{i=1}^{d} \pi_Y, i \sum_{j=1}^{d} \frac{v_j}{\lambda_Y} \log \left( \frac{v_j}{\lambda_Y} \right) = - \sum_{j=1}^{d} s_j \log s_j, \]
for \( s_j = \frac{\pi_Y}{\lambda_Y} \). But then, taking \( A_Y(\delta) = A_Y^{1+\delta} \), we also get \( H_Y(\delta) \approx -\sigma_Y^2 \delta \), for \( \delta \in \mathbb{R} \) small, and where \( \sigma_Y^2 > 0 \). This readily implies \( \Delta H_Y \Delta H > 0 \), and the proposition immediately follows since \( \Delta H_Y \Delta T < 0 \).

**B.3 Proof of Proposition 2**

This follows from Proposition 1 after noticing that for \( \delta \in \mathbb{R} \) small, the eigenvector \( v(\delta) \) corresponding to the perturbed matrix \( A^{1+\delta} \) satisfies
\[ v(\delta/2) \approx v(\delta) + v(0). \]
This implies that \( T(\frac{v(\delta)+v(0)}{2}) - T(0) \) has the same sign as \( \Delta T \) and the opposite sign as \( E(v(\delta),v(0)) \). Hence \( (T(\frac{v(\delta)+v(0)}{2}) - T(0)) \Delta H > 0 \) and \( E(v(\delta),v(0)) \Delta H < 0 \).

**B.4 Proof of Theorem 4**

We here provide a sketch of the main steps, and refer the reader to [21], Section 3, for more details.

We need to show that \( \Delta H \Delta R > 0 \). We first recall the definition of our robustness measure \( R \). Define the probability that the sample mean differs from the value \( \Phi \) by more than \( \epsilon \),
\[ P_\epsilon(n) = \mu \left\{ x \in \Omega : \left| \frac{1}{n} S_n \varphi(x) - \Phi \right| > \epsilon \right\}, \]
where the sample mean is given by,
\[ S_n \varphi(x) = \sum_{j=0}^{n-1} \varphi(\tau^j x) \]
\[ = \log a_{x_0 x_1} + \log a_{x_1 x_2} + \ldots + \log a_{x_{n-1} x_n} \]
\[ = \log a_{x_0 x_1} a_{x_1 x_2} \cdots a_{x_{n-1} x_n}, \]
and is such that \( \lim_{n \to \infty} S_n \varphi(x) = \int \varphi d\mu = \Phi \).

By the ergodic theorem, \( \lim_{n \to \infty} P_\epsilon(n) = 0 \); moreover, the convergence rate is at least exponentially fast, so that there exist constants, \( c_0, c_1 > 0 \), such that,
\[ \mu \left\{ x \in \Omega : \left| \frac{1}{n} S_n \varphi(x) - \Phi \right| > \epsilon \right\} \leq c_0 \exp^{-c_1 n}. \]
This motivates the robustness measure given by the fluctuation decay rate,
\[ R \equiv R_\epsilon = - \lim_{n \to \infty} \left[ \frac{1}{n} \log P_\epsilon(n) \right], \]
which characterizes the asymptotic value of the probability of the set of trajectories that deviate from the typical trajectory by \( \epsilon \) or less.
In order to better characterize $R$, consider the more general decay measures, 

$$
\mathcal{R}(\varphi, E) = -\frac{1}{n} \limsup_{n \to \infty} \left[ \mu \left\{ x \in \Omega : \frac{1}{n} \sum_{j=0}^{n-1} \varphi(\tau^j x) \in E \right\} \right]
$$

and

$$
\mathcal{R}(\varphi, E) = -\frac{1}{n} \liminf_{n \to \infty} \left[ \mu \left\{ x \in \Omega : \frac{1}{n} \sum_{j=0}^{n-1} \varphi(\tau^j x) \in E \right\} \right],
$$

where $E$ stands for arbitrary subsets of the real line. (Later we will be interested in sets of the form $E = \{ s \in \mathbb{R} : |s - \Phi| > \epsilon \}$.)

Next, one defines the function

$$
k_{\varphi}(s) = r - s - \sup_{\nu} \left\{ H_{\nu}(\tau) : \nu \text{ invariant under } \tau \text{ and } \int \varphi d\nu = s \right\}.
$$

Then we have,

$$
\mathcal{R}(\varphi, E) \geq -\inf \{ k_{\varphi}(s) : s \in E \} \text{ for every open set } E,
$$

and

$$
\mathcal{R}(\varphi, E) \leq -\inf \{ k_{\varphi}(s) : s \in E \} \text{ for every closed set } E.
$$

Moreover, $k_{\varphi}(s)$ is continuous and satisfies $k_{\varphi}(\Phi) = 0$ by the variational principle of equation (19). Hence,

$$
\mathcal{R} = \mathcal{R}(\varphi, E) = \mathcal{R}(\varphi, E) = -\inf \{ k_{\varphi}(s) : s \in E \} = -\min \{ k_{\varphi}(\Phi - \epsilon), k_{\varphi}(\Phi + \epsilon) \},
$$

and actually attains its minimum.

Now consider perturbations of the form $A^{1+\delta}$ corresponding to $\varphi(\delta) = (1 + \delta)\varphi$, where again, $\varphi(x) = \log a_{x_0 x_1}$. One can then define $\mathcal{R}(\delta)$ using $\varphi(\delta)$ instead of $\varphi$ and show that

$$
\mathcal{R}(\delta) = -\min \{ k_{\varphi(\delta)}(\Phi(\delta) - (1 + \delta)\epsilon), k_{\varphi(\delta)}(\Phi(\delta) + (1 + \delta)\epsilon) \},
$$

where $\lim_{n \to \infty} S_n \varphi(\delta)(x) = \int \varphi(\delta) d\mu(\delta) \equiv \Phi(\delta)$. Also, $H(\delta) = H_{\mu(\delta)}(\tau)$, where $\mu(\delta)$ is the measure corresponding to $\varphi(\delta)$.

Finally, one shows,

$$
k_{\varphi(\delta)}(\Phi(\delta) - (1 + \delta)\epsilon) = H(\delta) + (1 + \delta)\epsilon - \sup \left\{ H_{\nu}(\tau) : \nu \text{ invariant under } \tau \text{ and } \int \varphi d\nu = \Phi - \epsilon \right\}
$$

and, similarly,

$$
k_{\varphi(\delta)}(\Phi(\delta) + (1 + \delta)\epsilon) = H(\delta) - (1 + \delta)\epsilon - \sup \left\{ H_{\nu}(\tau) : \nu \text{ invariant under } \tau \text{ and } \int \varphi d\nu = \Phi + \epsilon \right\}.
$$

This readily implies that, from $\Delta \mathcal{R} = \mathcal{R}(\delta) - \mathcal{R}$ and $\Delta H = H(\delta) - H$, we have,

$$
\Delta H - \delta \epsilon \leq \Delta \mathcal{R} \leq \Delta H + \delta \epsilon,
$$

and hence (for $\Delta H$ bounded away from zero) we obtain $\Delta H \Delta \mathcal{R} > 0$, which completes the proof.